TWO APPROACHES TO INTEGRATION

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NAVAL POSTGRADUATE SCHOOL

Monterey, California

THESIS

TWO APPROACHES TO INTEGRATION

by

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Thesis Advisor: M.D. Weir

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and their integrals without any use of a theory of measure. We then use this theory of integrable functions in order to develop a theory of measurable functions and measurable sets. On the other hand, in the last chapter, we begin with an arbitrary measure space from which we construct, first a theory of measure in terms of set functions, and then a theory of measurable functions. Finally, we use this theory to develop a theory of integrable functions and their integrals.

The resulting classes of measurable functions and integrable functions from the two approaches are exactly the same.

Two Approaches to Integration

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ABSTRACT

This thesis investigates how to approach the theory of integration for utilization by the non-mathematician by exposing its essence and by considering integration theory from two different points of view. The first point of view is via linear functionals known as the Daniell development and the second point of view is via measure theory. In the first chapter, we develop a theory of measurable functions and their integrals without any use of a theory of measure. We then use this theory of integrable functions in order to develop a theory of measurable functions and measurable sets. On the other hand, in the last chapter, we begin with an arbitrary measure space from which we construct, first a theory of measure in terms of set functions, and then a theory of measurable functions. Finally, we use this theory to develop a theory of integrable functions and their integrals.

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I. INTRODUCTION

The definition given by Cauchy (1789-1857) is usually considered to be the first definition of an integral satisfying the modern requirements of rigor. Let f be a continuous function on the closed interval [a,b]. Cauchy considers the integral sum

$$S(P) = f(x_o)(x_1-x_o) + \dots + f(x_{n-1})(x_n-x_{n-1}) \qquad (1)$$

where $P = \{x_o, x_1, \dots, x_n\}$ is a partition of [a,b] (i.e., a sequence of points satisfying the inequalities $a = x_o < \dots < x_n = b$. The inclusion $P \subset P_1$ means that all of the points of the partition P are contained in the partition P_1. The sums S possess a "limit" as the diameter of P, denoted $d(P) \to 0$ which is called the definite integral $\int_a^b f$.

But with the growth of analysis the need to consider integrals of more irregular behaving functions became manifest. In analysis it is convenient to have a generalization of the integral which was considered in elementary Calculus and introduced in 1854 by the German mathematician Bernhard Riemann (1926-1866). Riemann generalized the integral for functions f defined and bounded over a closed interval [a,b]. He partitioned the interval into smaller subintervals having lengths $\Delta x_1, \dots, \Delta x_n$. Then he defined the oscillation of f in each Δx_i as the difference between the greatest and least

value of f in Δx_i. Next he proved that a necessary and sufficient condition for the sums $S = \sum_{i=1}^{N} f(x_i)\, \Delta x_i$, where x_i is <u>any</u> value of x in Δx_i, to approach a unique limit (so that the integral of f over [a,b] exists) as the maximum x_i tends to zero is that the <u>sum</u> of the intervals Δx_i in which the oscillation of f is greater than any given number ε must approach zero with the size of the intervals. This condition is always satisfied if f is continuous over [a,b]. Riemann then pointed out that this condition on the oscillations would allow one to replace "continuous function" by "continuous almost everywhere function" in the previous statement. (We will define the "almost everywhere" concept further on in the sequel, but intuitively it means that the set of points of discontinuity of f over [a,b] is a set that has no length, i.e., a set that does not contain any interval on the real line.)

However, the Riemann Integral lacks some very desirable properties. For example, consider the Dirichlet function f defined on the real line R as follows:

$$f(x) = \begin{cases} 1 \;, & \text{for x in [0.1] and x rational} \\ 0 \;, & \text{otherwise} \end{cases}$$

Next let $\{r_1, r_2, ...\}$ be an enumeration of the rational numbers in [0,1] and define for each k

$$f_k(x) = \begin{cases} 1 \ , & \text{if } x \text{ in } \{r_1,\ldots,r_k\} \\ 0 \ , & \text{if } x \text{ in } [0,1] - \{r_1,\ldots,r_k\} \end{cases}$$

It is easy to show that for all x in [0,1], $\lim_k f_k(x) = f(x)$. For each k, thefunction f_k is Riemann Integrable on [0,1], and $\int f_k = 0$, but f is not Reimann Integrable because we do not have a unique limit for the aforementioned Riemann sums. Thus, the nondecreasing sequence $\{f_k(x)\}$ converges to f(x) pointwise in [0,1], but the sequence of real numbers $\{\int_0^1 f_k\}$ does not converge to $\int_0^1 f$ (this latter integral does not even exist in the Riemann sense).

The above property is, however, possessed by the generalization of the Riemann integral introduced in 1904 by the French mathematician Henri Lebesque (1875-1941). Lebesque was a student of Felix Edouard Emile Borel and a Professor at the Collège de France. Guided by Borel's ideas and also by those of M. Jordan and G. Peano, he first presented his ideas on measure and the integral in his thesis, "Intégrale, longueur, aire." His work considerably improved on Borel's theory of measure and will be studied here.

Lebesque's Theory of Integration is based on his notion of "measure" of arbitrary sets of points on the real line which applies also to sets in n-dimensional Euclidean space. It turns out that a fucntion f is Riemann integrable on [a,b] whenever it is Lebesque Integrable there, but the converse is not necessarily true. The Dirichlet function f defined

above, for example, is not Riemann integrable over [0,1] but it is Lebesque Integrable there and, in this case $\int_0^1 f = 0$. Moreover, once the classical theory of Lebesque has been developed for functions from R^n to R it is an easy step to extend it to functions from R^n to C, where C is the complex number field.

In the second decade of this century, integration theory penetrated more and more into spaces differing from the initial prototype of n-dimensional Euclidean space. These departures from Eculidean spaces were dictated mainly by the developments in functional analysis. It was awkward to associate "integration" of functions over general spaces with the properties of the abstract elements and subsets of those spaces, and thus a new approach was called for. This new approach was expressed in the British-American mathematician D.J. Daniell's definition of the integral given in 1919. The method of development of Daniell is quite general and can be used to obtain many types of integrals. We will study his approach in the first chapter. In subsequent years, Daniell's integral underwent various modifications and a number of different versions of his construction are currently available.

The purpose of this thesis is to study these two approaches to integration theory, that of Lebesque and that of Daniell. In the first part, the development of the Daniell integral will begin with a vector lattice $\mathcal{S}$ of real-valued functions defined on a nonempty set X, together with a linear functional $\int$, called an <u>integral</u>, defined on $\mathcal{S}$. From that development,

we obtain the Daniell integral from which will follow the concept of "measure" for an arbitrary point set. In the second part, we will begin with the concept of this measure, develop its theoretical aspects, and then obtain an integral which is identical to the Daniell integral of the first part. Thus we will have come full circle.

II. FIRST APPROACH: THE DANIELL INTEGRAL

1. INTRODUCTION

THe development of this chapter begins with a vector lattice $\mathcal{S}$ of real-valued functions defined on a non-empty set X and a linear functional $\int$, called an integral and defined on $\mathcal{S}$, satisfying the following properties:

1) If f is in $\mathcal{S}$ and if $f > 0$, then $\int f \geq 0$.
2) If $\{f_n\}$ is a non-increasing sequence of nonnegative functions in $\mathcal{S}$ and $\lim f_k(x) = 0$ for all x in X, then $\lim_k \int f_k = 0$.

The vector lattice $\mathcal{S}$ in this development takes the role of the vector lattice of all Riemann integrable functions whenever the non-empty set X is some n-dimensional Euclidean space R^n.

Through the concept of a "null set" we will extend the vector lattice of Daniell Integrable functions from $\mathcal{S}$ to $\mathcal{L}$, the class of all Lebesque Integrable functions whenever X is R^n. Next, we extend $\mathcal{L}$ to $\tilde{\mathcal{M}}$ a vector lattice which is the algebra of "measurable functions" in such a way that every sequence $\{f_k\}$ of functions in $\tilde{\mathcal{M}}$ converges a.e. to a limit function f belonging to $\tilde{\mathcal{M}}$. However, the class $\mathcal{L}$ does not have that convergence property. Finally, from the notion of a measurable function, we obtain the concept of measure for an arbitrary point set.

2. VECTOR LATTICES

To begin, we need some basic definitions and elementary facts that will be used throughout the thesis.

A metric space (X,d) is a non-empty set X together with a metric d defined on X such that:

$$d: X \times X \to R^+$$

and for all x, y, z in X the following properties hold:

1) $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$;
2) $d(x,y) = d(y,x)$;
3) $d(x,y) \leq d(x,z) + d(z,y)$.

Properties (1), (2) and (3) are called, respectively, the positive definite, symmetric, and transitive properties.

The diameter of a set E in a metric space (X,d), denoted by $d(E)$ is defined as follows:

$$d(E) = \sup\{d(x,y): x,y \text{ in } E\}$$

The diameter of a set may be finite or infinite. If $d(E)$ is finite, then E is said to be bounded.

The distance between two sets E and F, denoted by $d(E,F)$ is defined as follows:

$$d(E,F) = \inf\{d(x,y): x \text{ in } E \text{ and } y \text{ in } F\} .$$

For any pont x in a metric space (X,d) and any positive real number r, the _open ball_ with center x and radius r is defined to be the set:

$$B(x;r) = \{y: d(x,y) < r\} .$$

If E is any set in X, we define:

x is an _interior point_ of E if some open ball with center x is contained in E;

x is a _boundary point_ of E if every open ball with center x contains at least one point of E and at least one point of -E, the complement of E.

x is an _exterior point_ of E if some open ball with center x is contained in -E.

The set of all interior, boundary and exterior points of E are called, respectively, the _interior_, _boundary_, and _exterior_ of E and are denoted by Int(E), Bdy(E) and Ext(E).

Clearly, Int(E), Bdy(E) and Ext(E), constitue a partition of X; that is, they are pairwise disjoint and

$$X = \mathrm{Int}(E) \bigcup \mathrm{Bdy}(E) \bigcup \mathrm{Ext}(E) .$$

If E is a set such that E = Int(E), then E is said to be _open_.

A set E is said to be _closed_ if its complement is open. If E is closed, then -E = Ext(E) or, $E = \mathrm{Int}(E) \bigcup \mathrm{Bdy}(E)$.

Thus, a set is closed if and only if it contains all of its boundary points.

Any open set which contains A is called an open neighborhood of A. Any set containing an open neighborhood of A is called a neighborhood of A. Clearly, an open ball with center x is an open neighborhood of x. Also, if N is any neighborhood of x, then there exists an open ball $B(x;r)$ such that $B(x;r) \subset N$.

A point b in X is the limit of the sequence of points $\{x_n\}$ in the metric space (X,d), written $\lim_{n\to\infty} x_n = b$, or simply $\lim x_n = b$, if for each $\varepsilon > 0$ there exists a positive integer n_o such that $d(x_n,b) < \varepsilon$ whenever $n \geq n_o$. It is also said that x_n converges to b.

Let (X,d_1) and (Y,d_2) be two metric spaces. A function f from X to Y is said to be continuous if for every open set G in the range space, $f^{-1}(G)$ is open in the domain, where $f^{-1}(G) = \{x: f(x) \text{ is in } G\}$.

A vector space with a multiplication property related to the vector space operations is called an algebra. Let us be more precise.

An algebra is a linear space X with an operation, called multiplication, from $X \times X$ into X which has the following properties for all x, y, z in X and a in the field F:

1) $x(yz) = (xz)z$;

2) $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;

3) $a(xy) = (ax)y = x(ay)$

If the multiplication is commutative, then the algebra is called a Commutative algebra. An algebra X is called an algebra with unit (identity) if there exists a nonzero element in X, denoted by e and called the multiplicative unit or identity, such that for all x in X

$$xe = x = ex \quad .$$

A relation $\mathfrak{R}$ in a set A is called a partial order in A if:

1) $a \mathfrak{R} a$ for all a in A;
2) $a \mathfrak{R} b$ and $b \mathfrak{R} a$ imply $a = b$; and
3) $a \mathfrak{R} b$ and $b \mathfrak{R} c$ imply $a \mathfrak{R} c$.

A non-empty set A with a partial order defined in it is called a partially ordered set.

Let f be a real-valued function defined on a set X, and let A be a non-empty subset of X. The the least upper bound (respectively, greatest lower bound) of the set f(A) in R* is called the least upper bound (respectively, greatest lower bound) of f in A, and is denoted by $\sup_{x \varepsilon A} f(x)$ (respectively, $\inf_{x \varepsilon A} f(x)$).

The number $a = \sup_{x \varepsilon A} f(x)$ is characterized by the following two properties:

1) For all x in A, $f(x) \leq a$
2) For every $b < a$, there exists an x in A such that $b < f(x) \leq a$.

It can be shown that

$$\inf_{x \varepsilon A} f(x) = - \sup_{x \varepsilon A}(-f(x)) .$$

A <u>lattice</u> is a partially ordered set in which any two elements have a least upper bound and a greatest lower bound.

The least upper bound of the elements a and b is denoted by $a \vee b$ (a sup b) and the greatest lower bound by $a \wedge b$ (a inf b).

A set which is both a linear space (i.e., vector space) and a lattice is called a <u>vector lattice</u>.

Let $\mathcal{S}$ be a vector lattice of real-valued functions defined on a non-empty arbitrary set X. Let us consider a linear functional $\int$, called an integral, defined on $\mathcal{S}$ and satisfying the following two properties:

1) If f is in $\mathcal{S}$ and $f > 0$, then $\int f \geq 0$.
2) If $\{f_n\}$ is a non-increasing sequence of nonnegative functions in $\mathcal{S}$ and $\lim f_k(x) = 0$ for all x in X, then $\lim \int f_k = 0$.

If for g and h in $\mathcal{S}$, we define

$$(g \wedge h)(x) = \min\{g(x), h(x)\}$$

$$(g \vee h)(x) = \max\{g(x), h(x)\}$$

then $g \wedge h$ and $g \vee h$ are the greatest lower bound and least upper bound, respectively, of g and h.

Also, the following formulas hold:

$$g \wedge h = \frac{1}{2} (g + h - |g-h|) \tag{1}$$

$$g \vee h = \frac{1}{2} (g + h + |g-h|) \tag{2}$$

To prove (1), there are two cases to be considered:

a) If $g \geq h$, then the left hand side gives

$$\begin{aligned}(g \wedge h)(x) &= \min\{g(x), h(x)\} \\ &= h(x), \text{ for every } x \text{ in } X\end{aligned}$$

and the right hand side gives

$$\begin{aligned}\frac{1}{2}(g + h - |g-h|)(x) &= \frac{1}{2}(g(x) + h(x) - (g(x)-h(x))) \\ &= \frac{1}{2}\, 2h(x) \\ &= h(x), \text{ for every } x \text{ in } X.\end{aligned}$$

b) If $g < h$, then for all x in X

$$\begin{aligned}(g \wedge h)(x) &= \min\{g(x), h(x)\} \\ &= g(x), \text{ for every } x \text{ in } X.\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{1}{2}(g + h - |g-h|)(x) &= \frac{1}{2}(g(x) + h(x) - (h(x)-g(x))) \\ &= \frac{1}{2}\, 2g(x) \\ &= g(x) \quad \text{for every } x \text{ in } X\end{aligned}$$

which proves (1).

By a similar argument, we obtain (2).

It is not difficult to show that the following properties hold:

If f, g and h are in $\mathcal{S}$, then

$$f \wedge g = -((-f) \vee (-g)) \tag{3}$$

$$f \vee g = -((-f) \wedge (-g)) \tag{4}$$

$$f \vee (g \wedge h) = (f \vee g) \wedge (f \vee h) \tag{5}$$

$$(f \vee g) \wedge h = (f \wedge h) \vee (g \wedge h) \tag{6}$$

Also the following result is an immediate consequence of the definition:

If $\lim g_k = g$ and $\lim h_k = h$, then

$\lim(g_k \wedge h_k) = g \wedge h$ and $\lim(g_k \vee h_k) = g \vee h$.

Now, since $\mathcal{S}$ is a vector lattice, for each g in $\mathcal{S}$, we define

$$g^+ = g \vee 0 \quad \text{and} \quad g^- = -(g \wedge 0)$$

Then g^+ and g^- belong to $\mathcal{S}$. Observe that $g^+ \geq 0$ and $g^- \geq 0$. By an above argument

$$g^+ = \frac{1}{2}(|g| + g)$$

and

$$g^- = \frac{1}{2}(|g| - g).$$

From these results, we obtain the following formulas:

$$g = g^+ - g^- \quad \text{and} \quad |g| = g^+ + g^- .$$

Let f be an extended real-valued function on X, and let a be any real number. We define the functions af , $a+f$, $|f|^a$, respectively for each x in X by the formulas $af(x)$, $a+f(x)$, $|f(x)|^a$.

3. NULL SET

For the integral there are certain sets which are negligible in some sense. Sets which will be negligible with respect to Daniell Integration can be characterized in terms of functions in $\mathcal{S}$ as we now define.

3.1. DEFINITION. A set E in X is said to be a null set if there is a nondecreasing sequence $\{g_k\}$ of nonnegative functions in $\mathcal{S}$ such that $\{g_k(x)\}$ diverges to ∞ for each x in E and $\{\int g_k\}$ converges.

If a proposition P(x) holds for all x except for a null set, then we say that P(x) holds Almost everywhere (abbreviated: for a.a.x).

The following result is very useful.

3.2. PROPOSITION. The countable union of null sets is a null set.

PROOF: Let $\{E_k\}_{k=1}^{\infty}$ be a Countable Collection of null sets. Then for each k, there exists a nondecreasing sequence $\{g_{kj}\}_{j=1}^{\infty}$ of nonnegative functions in $\mathcal{S}$ such that $\{g_{kj}(x)\}$ diverges to ∞ for each x in E_k and $\{\int g_{kj}\}_{j=1}^{\infty}$ converges.

Let $M_k = \lim_{j\to\infty} \int g_{kj}$. By property (1) of $\mathcal{S}$ notice that $\{\int g_{kj}\}$ is a nondecreasing sequence of nonnegative real numbers. Hence, for each k, there exists n(k) such that

$$M_k - \int g_{kj} < 2^{-k} \quad \text{if} \quad j \geq n(k) \quad .$$

Define the sequence $\{\hat{g}_{kj}\}$ where $\hat{g}_{kj} = g_{k,n(k)+j-1}$. Clearly, $\{\hat{g}_{kj}\}$ is a nondecreasing sequence of nonnegative functions in $\mathcal{S}$ such that $\{\hat{g}_{kj}(x)\}$ diverges to ∞ for each x in E_k and $\{\int \hat{g}_{kj}\}_{j=1}^{\infty}$ converges (monotonically) to M_k. Also for each k

$$M_k - \int \hat{g}_{kj} < 2^{-k} \quad \text{for all } j = 1,2,3,\ldots$$

Next, for each k define the sequence $\{h_{kj}\}_{j=1}^{\infty}$ where

$$h_{kj} = \hat{g}_{kj} - \hat{g}_{k,1} = g_{k,n(k)+j-1} - g_{k,n(k)} \ .$$

Then $\{h_{kj}\}_{j=1}^{\infty}$ is a nondecreasing sequence of nonnegative functions in $\mathcal{S}$. Moreover $\{h_{kj}(x)\}_{j=1}^{\infty}$ diverges to ∞ for each x in E_k (because g_{kj} does), and for every j,

$$\int h_{kj} = \int \hat{g}_{kj} - \int \hat{g}_{k1} \leq M_k - \int \hat{g}_{k1} < 2^{-k}$$

therefore, the sequence $\{ \int h_{kj}\}_{j=1}^{\infty}$ converges.

Finally, define the sequence

$$h_m = \sum_{j=1}^{m} h_{jm}$$

Observe that for each m, h_m is in $\mathcal{S}$.

CLAIM: $\{h_m(x)\}$ diverges to ∞ for each x in $E = \bigcup_{k=1}^{\infty} E_k$.

Let x in E so that x is in E_k for some k. Then $\{h_{kj}(x)\}$ diverges to ∞ so that for every positive $r > 0$, there exists n(r) such that $h_{kj}(x) > r$ if $j \geq n(r)$. Now choose m such that $m > \max\{k,n(r)\}$. Then

$$h_m(x) = \sum_{j=1}^{m} h_{jm}(x) \geq h_{k,n(r)}(x) > r .$$

CLAIM: The sequence $\{\int h_m\}$ converges. For each m,

$$\int h_m = \int \sum_{j=1}^{m} h_{jm} = \sum_{j=1}^{m} \int h_{jm}$$

Since integral is a linear functional.

So

$$\int h_m = \sum_{j=1}^{m} \int h_{jm} < \sum_{j=1}^{m} 2^{-j} < \sum_{j=1}^{\infty} 2^{-j} = 1 .$$

Also, since $\{h_{kj}\}_{j=1}^{\infty}$ is a nondecreasing sequence of non-negative functions, $\{h_m\}$ is also a nondecreasing sequence

of nonnegative functions. Thus, $\{\int h_m\}$ is a nondecreasing sequence of nonnegative real numbers that is bounded above; hence converges. Therefore E is a null set.

∎

Next, we will show that the limit of the integrals of a sequence of fucntions that approach zero almost everywhere is itself zero.

3.3. <u>PROPOSITION</u>. <u>If</u> $\{f_n\}$ <u>is a nonincreasing sequence of nonnegative functions in</u> $\mathcal{S}$ <u>and</u> $\lim f_x(x) = 0$ <u>for</u> a.a.x, <u>then</u> $\lim \int f_k = 0$.

<u>PROOF</u>: For each x in X, $\{f_k(x)\}$ is a nonincreasing sequence bounded below by zero, and hence converges. Thus, let $f(x) = \lim f_k(x) = 0$ for a.a.x as in the hypothesis. Set $E = \{x \text{ in } E: f(x) \neq 0\}$, so that E is a null set. Thus there exists a nondecreasing sequence $\{g_k\}$ of nonnegative functions in $\mathcal{S}$ such that $\{g_k(x)\}$ diverges to ∞ for c in R and $\{\int g_k\}$ converges. Let $c = \lim \int g_k$. We consider two cases: $c > 0$ and $c = 0$ (c should not be negative by property (1) of $\mathcal{S}$ in Section 2).

If $c > 0$, let $\varepsilon > 0$ be arbitrary and set $\hat{g}_k = \frac{\varepsilon}{2c} g_k$. Then $\{\hat{g}_k\}$ is a nondecreasing sequence of nonnegative functions in $\mathcal{S}$. Define $h_k = f_k - \hat{g}_k$. Since $f_{k+1} \leq f_k$ and $-\hat{g}_{k+1} < -\hat{g}_k$ so $f_{k+1} - \hat{g}_{k+1} \leq f_k - \hat{g}_k$. Thus, $\{h_k\}$ is a nonincreasing sequence of functions in $\mathcal{S}$ because $\mathcal{S}$ is a real linear space. Also, $\{h_k^+\}$ is a nonincreasing sequence of nonnegative functions in $\mathcal{S}$. Now,

$$h_k^+ = (f_k - \hat{g}_k)^+ = (f_k - \hat{g}_k) \vee 0$$

For if x in E, then $\lim h_k^+(x) = 0$ since

$$\lim(-g_k(x)) = -\infty$$

If x in X-E, then

$$\lim h_k^+(x) = [\lim f_k(x) - \lim \hat{g}_k(x)] \vee 0$$
$$= -\lim \hat{g}_k(x) \vee 0 = 0$$

since $\hat{g}_k$ is a nondecreasing sequence of nonnegative functions.

Thus, $\lim(h_k^+)(x) = 0$ for every x in X. According to Rule (2) of $\mathcal{S}$ in Section 2, then

$$\lim \int h_k^+ = 0$$

Since $h_k \leq h_k^+$ it follows that $\lim \int h_k \leq 0$.

Hence $\lim \int (f_k - \hat{g}_k) \leq 0$

or $\lim \int f_k \leq \lim \int \hat{g}_k = \lim \int \frac{\varepsilon}{2c} g_k = \frac{\varepsilon}{2}$

because integral is a linear functional. Also $0 \leq f_k$ for every k, so we have

$$0 \leq \lim \int f_k < \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we conclude $\lim \int f_k = 0$.

Finally, if $c = 0$, set $h_k = f_k - g_k$ and repeat the above procedure. Then we obtain, as before

$$\lim_k \int (f_k - g_k) \le 0$$

or
$$0 \le \lim \int f_k \le \lim \int g_k = 0$$

and the conclusion still holds.

The following proposition provides us with the key theorem needed to extend the definition of the integral to a class of functions larger than the vector lattice $\mathcal{S}$.

3.4. PROPOSITION. *If $\{g_k\}$ and $\{h_k\}$ are nondecreasing sequences of functions in $\mathcal{S}$ such that* $\lim g_k(x) \le \lim h_k(x)$ *for* a.a.x *and* $\{\int g_k\}$ *and* $\{\int h_k\}$ *are bounded, then*

$$\lim \int g_k \le \lim \int h_k$$

PROOF: For a fixed positive i, $\{g_i - h_k\}$ is a nonincreasing sequence of functions in $\mathcal{S}$ since $\{h_k\}$ is a nondecreasing sequence of functions in $\mathcal{S}$. Since $\mathcal{S}$ is a vector lattice, $(g_i - h_k)^+ = (g_i - h_k) \vee 0$ also belongs to $\mathcal{S}$. Since $\{g_k\}$ and $\{h_k\}$ are nondecreasing and $\lim g_k \le \lim h_k$ a.e., for a.a.x, there exists $k_o(x)$ such that for $k > k_o(x)$, $g_i(x) \le h_k(x) \le \lim h_k(x) = h(x)$. Thus, for every $k \ge k_o(x)$ $g_i(x) - h_k(x) \le 0$, then $[g_i(x) - h_k(x)]^+ = 0$.
Hence, $\lim_k [g_i(x) - h_k(x)]^+ = 0$ for a.a.x.

By the previous proposition 3.3, we have

$$\lim_{k} \int (g_i - h_k)^+ = 0$$

Since $g_i - h_k \quad (g_i - h_k)^+$ implies $\lim\limits_{k} \int (g_i - h_k) \leq 0$

Therefore, $\int g_i \leq \lim \int h_k$ for every i. Since this inequality holds for every integer i, thus

$$\lim \int g_k \leq \lim \int h_k$$

∎

4. EXTENSION OF THE INTEGRAL TO $\tilde{\mathcal{S}}$

The development of the preceding section will now be extended from the integral of functions in $\mathcal{S}$ to a larger class of functions. This development will be accomplished in two stages and the material in this section represents the first stage.

4.1. <u>DEFINITION</u>. Let $\tilde{\mathcal{S}}$ be the set of real-valued functions g defined on X for which there is a nondecreasing sequence $\{g_k\}$ of functions in $\mathcal{S}$ such that $\lim g_k(x) = g(x)$ for a.a.x and the sequence $\{\int g_k\}$ is bounded. We define

$$\int g = \lim \int g_k \ .$$

Since $\{\int g_k\}$ is a bounded nondecreasing sequence of real numbers, then it converges. Next, because Proposition 3.4,

the definition of $\int g$ is independent of the particular sequence $\{g_k\}$ of functions in $\mathcal{S}$. Therefore the definition of $\int g$ is well defined.

4.2. PROPOSITION. *If g and h are in $\tilde{\mathcal{S}}$ and a is a nonnegative real number, then $g+h$ and ag are in $\tilde{\mathcal{S}}$ and*

$$\int (g+h) = \int g + \int h \quad \text{and} \quad \int ag = a \int g \ .$$

PROOF: Let g and h be in $\tilde{\mathcal{S}}$, then there are two nondecreasing sequences $\{g_k\}$ and $\{h_k\}$ of functions in $\mathcal{S}$ such that $\lim g_k(x) = g(x)$ and $\lim h_k(x) = h(x)$ for a.a.x , and the sequences $\{\int g_k\}$ and $\{\int h_k\}$ are bounded. Moreover

$$\int h = \lim \int h_k \quad \text{and} \quad \int g = \lim \int g_k$$

The sequence $\{g_k + h_k\}$ is a nondecreasing sequence of functions in $\mathcal{S}$ since $\{g_k\}$ and $\{h_k\}$ are so, and $\mathcal{S}$ is a vector lattice. Also

$$\begin{aligned} \lim(g_k + h_k)(x) &= \lim g_k(x) + \lim h_k(x) \quad \text{for a.a.x} \\ &= g(x) + h(x) \quad \text{for a.a.x} \end{aligned}$$

Because the integral on $\mathcal{S}$ is a linear functional it follows that

$$\int (g_k + h_k) = \int g_k + \int h_k$$

Hence the sequence $\{\int (g_k + h_k)\}$ is bounded due to the bounded properties of $\{\int g_k\}$ and $\{\int h_k\}$. For that reason we have that $g + h$ belongs to $\tilde{\mathcal{S}}$, and that

$$\int (g + h) = \lim \int (g_k + h_k) = \lim (\int g_k + \int h_k)$$

$$= \lim \int g_k + \lim \int h_k$$

$$= \int g + \int h .$$

Next, if a is a nonnegative real number then $\{ag_k\}$ is a nondecreasing sequence of functions in $\mathcal{S}$ and

$$\lim (ag_k) = a \lim g_k = ag \quad \text{for a.a.x}$$

Since $\{\int g_k\}$ is bounded then $\{a \int g_k\} = \{\int ag_k\}$ is bounded. This shows that ag is in $\tilde{\mathcal{S}}$ and

$$\int ag = \lim \int ag_k = a \lim \int g_k = a \int g .$$

∎

4.3. <u>PROPOSITION</u>. <u>The class of $\tilde{\mathcal{S}}$ forms a lattice</u>.

PROOF: Let g and h be in $\tilde{\mathcal{S}}$. Then there are two nondecreasing sequences $\{g_k\}$ and $\{h_k\}$ of functions in $\mathcal{S}$ such that $\{\int g_k\}$ and $\{\int h_k\}$ are bounded and $\{g_k\}$ and $\{h_k\}$ converge almost everywhere to g and h, respectively. We will show that $g \wedge h$ is in $\tilde{\mathcal{S}}$. The proof that $g \vee h$ is in $\tilde{\mathcal{S}}$ is similar.

Clearly, $\{g_k \wedge h_k\}$ is a nondecreasing sequence of functions in $\mathcal{S}$. Since $g_k(x) \wedge h_k(x) \leq g_k(x)$ then $\int g_k(x) \wedge h_k(x) \leq \int g_k(x)$

by Proposition 3.4. Hence, $\{\int(g_k \wedge h_k)\}$ is bounded.

By (7) of Section 2, we have

$$\begin{aligned}\lim(g_k \wedge h_k) &= \lim g_k \wedge \lim h_k \\ &= g \wedge h \quad \text{for a.a.x}\end{aligned}$$

Thus, by definition $g \wedge h$ is in $\tilde{\mathcal{S}}$.

4.4. PROPOSITION. *If* g *and* h *in* $\tilde{\mathcal{S}}$ *and* $g(x) \leq h(x)$ *for* a.a.x, *then*

$$\int g \leq \int h$$

PROOF: Let g and h be in $\tilde{\mathcal{S}}$ and $g(x) \leq h(x)$ for a.a.x and let $\{g_k\}$ and $\{h_k\}$ be nondecreasing sequences of functions in $\mathcal{S}$ such that $\{\int g_k\}$ and $\{\int h_k\}$ are bounded and $\lim g_k(x) = g(x)$ and $\lim h_k(x) = h(x)$ for a.a.x, by definition we have

$$\int g = \lim \int g_k \quad \text{and} \quad \int h = \lim \int h_k .$$

Since $g(x) \leq h(x)$ for a.a.x then $\lim g_k(x) \leq \lim h_k(x)$ for a.a.x. By proposition 3.4, we have

$$\int g = \lim \int g_k \leq \lim \int h_k = \int h$$

Let g be any element in $\mathscr{S}$. Then g is a limit of a nondecreasing sequence $\{g_k\}$ of functions in $\mathscr{S}$ where $g(x) = g_k(x)$ for every x in X and every k. The sequence $\{g_k\}$ satisfies the requirement of definition 4.1. Thus g is in $\tilde{\mathscr{S}}$. If g is in $\mathscr{S}$, it is obvious that its integral as an element of $\tilde{\mathscr{S}}$ agrees with its integral as an element of $\mathscr{S}$.

Also, if g is in $\tilde{\mathscr{S}}$ and $g = h$ almost everywhere, we will show that h is in $\tilde{\mathscr{S}}$ and $\int g = \int h$. Indeed, let $\{g_k\}$ be a nondecreasing sequence of functions in $\mathscr{S}$ such that $\lim g_k(x) = g(x)$ for a.a.x. Let E_1 be a null set on which $\lim g_k(x) = g(x)$ is false. Now define $f_k(x) = g_k(x)$ and $f(x) = g(x)$ if x is in $X-E_2$ and $f_k = f(x)$ for x in E_2, where E_2 is a null set. Thus $\{f_k\}$ is a nondecreasing sequence of functions in $\mathscr{S}$ according to definition 4.1. This implies that f is in $\tilde{\mathscr{S}}$ and $\lim \int f_k = \int f$. Morever, then $\int f = \int g$.

In general if g is in $\tilde{\mathscr{S}}$, we can not say that -g is in $\tilde{\mathscr{S}}$ because the sequence $\{g_k\}$ defined in definition 4.1 such that $\lim g_k(x) = g(x)$ for a.a.x, gives $\lim_k(-g_k(x)) = -g(x)$ for a.a.x. However, the sequence $\{-g_k\}$ is no longer a sequence of functions satisfying the requirements in definition 4.1. Thus $\tilde{\mathscr{S}}$ is not a linear space. At the next stage we will embed $\tilde{\mathscr{S}}$ in a space that is a linear space.

The following is the basic convergence result for $\tilde{\mathscr{S}}$.

4.5. PROPOSITION <u>Let</u> $\{g_k\}$ <u>be a nondecreasing sequence of functions in</u> $\tilde{\mathcal{S}}$ <u>such that</u> $\{\int g_k\}$ <u>is bounded</u>. <u>Then</u> $\{g_k(x)\}$ <u>converges for</u> a.a.x <u>and if</u> $g(x) = \lim_k g_k(x)$ <u>for</u> a.a.x <u>then</u> g <u>is in</u> $\tilde{\mathcal{S}}$ <u>and</u>

$$\int g = \lim \int g_k$$

PROOF: For each positive integer k let $\{g_{kj}\}_{j=1}^{\infty}$ be the sequence of functions required by the definition for $\tilde{\mathcal{S}}$ giving the following tableau:

$$\begin{array}{cccccccc}
g_{11} & g_{12} & \cdots & g_{1m} & \cdots & g_{1n} & \rightarrow & g_1 \\
g_{21} & g_{22} & \cdots & g_{2m} & \cdots & g_{2n} & \rightarrow & g_2 \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
g_{m1} & g_{m2} & \cdots & g_{mm} & \cdots & g_{mn} & \rightarrow & g_m \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots &
\end{array}$$

If we define $h_m = g_{1m} \vee g_{2m} \vee \ldots \vee g_{mm}$, then $\{h_m\}$ is a nondecreasing sequence of functions in $\mathcal{S}$. Since for all j, $g_{kj}(x) \leq g_k(x)$ for a.a.x it follows that

(4.6) $$h_m(x) \leq g_1(x) \vee g_2(x) \vee \ldots \vee g_m(x) = g_m(x)$$

for a.a.x and, therefore, $\int h_m \leq \int g_m$.

This shows that $\{\int h_m\}$ is bounded and, hence, converges. From the definition of a null set on which $\{h_m\}$ diverges to ∞, $\{h_m\}$ must then converge almost everywhere (except on the null set).

If we let h be a function such that $h(x) = \lim_m h_m(x)$ for a.a.x, then h is in $\tilde{\mathcal{S}}$ and $\int h = \lim \int h_m$.

For each k and for $m \geq k$ we have:

$$g_{km}(x) \leq h_m(x) \qquad \text{for all } x$$

and therefore taking the limit with respect to m, we obtain

$$\lim_m g_{km} = g_k(x) \leq \lim_m h_m(x) = h(x) \text{ for a.a.x.}$$

Thus, $\{g_k\}$ is a nondecreasing sequence. It follows that the limit with respect to k exists for a.a.x.

Moreover, $\lim g_k(x) = g(x) \leq h(x)$ for a.a.x. Also by (4.6) we have $h(x) \leq g(x)$ for a.a.x. Thus g is in $\tilde{\mathcal{S}}$ and $\int g = \int h$. Since

$$h_m(x) \leq g_m(x) \leq h(x) \qquad \text{for a.a.x,}$$

propoaition 4.4 implies that $\int h_m \leq \int g_m \leq \int h$ for all k

$$\int h = \lim \int h_m \leq \lim \int g_m = \int h$$

Thus $\int g = \lim \int g_k$

completing the proof.

The following result is an immediate consequence of Proposition 4.5 and is sometimes more convenient to apply.

4.7. COROLLARY. *Let $\{h_k\}$ be a sequence of nonnegative functions in $\tilde{\mathcal{S}}$ such that $\{\int \sum_{k=1}^{m} h_k\}$ is bounded. Then $\sum_{k=1}^{\infty} h_k(x)$ converges for* a.a.x *and if* $g(x) = \sum_{k=1}^{\infty} h_k(x)$ *for* a.a.x, *then* g *is in* $\tilde{\mathcal{S}}$ *and* $\int g = \sum_{k=1}^{\infty} \int h_k$.

PROOF: Let $g_m = \sum_{k=1}^{m} h_k$. Since $\{h_k\}$ is a sequence of nonnegative functions in $\tilde{\mathcal{S}}$ then $\{g_m\}$ is a sequence of nondecreasing functions in $\tilde{\mathcal{S}}$.

Now $\{\int g_m\} = \{\int \sum_{k=1}^{m} h_k\}$ so $\{\int g_m\}$ is bounded since $\{\sum_{k=1}^{m} \int h_k$ is bounded. By Proposition 4.5, $\{g_m(x)\}$ converges for a.a.x. Thus, $\sum_{k=1}^{\infty} h_k(x)$ converges for a.a.x.

Let $g(x) = \lim g_m(x) = \sum_{k=1}^{\infty} h_k(x)$ for a.a.x. By (4.5) g is in $\tilde{\mathcal{S}}$ and

$$\int g = \lim \int g_m = \lim_m \int \sum_{k=1}^{m} h_k$$

$$= \lim_m \sum_{k=1}^{m} \int h_k$$

$$= \sum_{k=1}^{\infty} \int h_k \ .$$

5. THE DANIELL INTEGRAL

Since the difference of two functions in $\tilde{\mathcal{S}}$ need not be in $\tilde{\mathcal{S}}$, the class $\tilde{\mathcal{S}}$ is not generally a linear space. We next define a set which contains $\tilde{\mathcal{S}}$ and is also a linear space.

5.1. DEFINITION. Let $\mathcal{L}$ be the set of all real-valued functions f defined on the nonempty set X such that $f = g - h$ a.e. where g and h are in $\tilde{\mathcal{S}}$.

The set $\mathcal{L}$ is called the set of Daniell Integrable functions on X and the Daniell Integral of f is defined as follows:

$$\int f = \int g - \int h$$

To justify the above definition of the integral, suppose $f = g - h$ a.e. and $f = g_1 - h_1$ a.e. where g, h, g_1 and h_1 are in $\tilde{\mathcal{S}}$. Then

$$g + h_1 = g_1 + h \quad \text{a.e.}$$

Therefore $\int g + \int h_1 = \int (g + h_1) = \int (g_1 + h) = \int g_1 + \int h$

Then $$\int g - \int h = \int g_1 - \int h_1$$

Note that if f_1 is in $\mathcal{L}$ and $f_2 = f_1$ a.e. then f_2 is in $\mathcal{L}$ and $\int f_1 = \int f_2$. Also, $\tilde{\mathcal{S}}$ is a subset of $\mathcal{L}$. If f is in $\tilde{\mathcal{S}}$, then its Daniell integral clearly agrees with its integral as an element of $\tilde{\mathcal{S}}$.

We now consider the basic properties of $\mathscr{L}$ as well as the integral over $\mathscr{L}$.

5.2. <u>THEOREM</u>. <u>The set $\mathscr{L}$ is a linear space and the integral is a linear functional on $\mathscr{L}$.</u>

PROOF. Let g and f be in $\mathscr{L}$, and let a be an arbitrary real number. We will show $f + g$ and af belong to $\mathscr{L}$ and that

$$\int (f+g) = \int f + \int g \quad \text{and} \quad \int af = a \int f .$$

Since f and g are in $\mathscr{L}$ we have $f = f_1 - f_2$ and $g = g_1 - g_2$ a.e., where f_1, f_2, g_1 and g_2 are in $\tilde{\mathscr{S}}$, and $\int f = \int f_1 - \int f_2$ and $\int g = \int g_1 - \int g_2$.

Now, $f+g = (f_1+g_1) - (f_2+g_2)$ a.e.

By Proposition 4.2, it follows that f_1+g_1 and f_2+g_2 are in $\tilde{\mathscr{S}}$. Therefore f+g is in $\mathscr{L}$.

Also

$$\begin{aligned} \int (f+g) &= \int (f_1+g_1) - \int (f_2+g_2) \\ &= \int f_1 + \int g_1 - (\int f_2 + \int g_2) \\ &= (\int f_1 - \int f_2) + (\int g_1 - \int g_2) \\ &= \int f + \int g \end{aligned}$$

Likewise, $af = a(f_1-f_2) = af_1 - af_2$ a.e.

If $a > 0$, then af_1 and af_2 are in $\tilde{\mathscr{S}}$. Thus af is in $\mathscr{L}$.

If $a < 0$, then $(-a)f_1$ and $(-a)f_2$ are in $\tilde{\mathcal{S}}$, and $af = af_1 - af_2 = (-a)f_2 - (-a)f_1$ a.e.

Thus in either case af is in $\mathcal{L}$.

Also for $a > 0$

$$\int af = \int a(f_1-f_2) = \int (af_1 - af_2) = \int af_1 - \int af_2$$

$$= a \int f_1 - a \int f_2 = a \left(\int f_1 - \int f_2\right)$$

$$= a \int f .$$

Similarly for $a < 0$

$$\int af = \int (-af_2 - (-a)f_1) = \int (-a)f_2 - \int (-a)f_1$$

$$= -a \int f_2 + a \int f_1 = a\left(\int f_1 - \int f_2\right)$$

$$= a \int f .$$

and the proof is complete.

5.3. <u>THEOREM</u>. <u>The space $\mathcal{L}$ is a lattice</u>.

PROOF. Take g and f in $\mathcal{L}$, and suppose $f = f_1-f_2$ a.e., where f_1 and f_2 are in $\tilde{\mathcal{S}}$. Then $f_1 \wedge f_2$ and $f_1 \vee f_2$ are in $\tilde{\mathcal{S}}$ since $\tilde{\mathcal{S}}$ is a lattice (Proposition 4.3). Recall that

$$f_1 \wedge f_2 = \frac{1}{2} (f_1 + f_2 - |f_1-f_2|)$$

and $$f_1 \vee f_2 = \frac{1}{2} (f_1 + f_2 + |f_1-f_2|)$$

Hence, we have $|f| = |f_1 - f_2| = (f_1 \vee f_2) - (f_1 \wedge f_2)$ a.e., therefore, $|f|$ is in $\mathscr{L}$. From this result and Theorem 5.2, we have

$$f \wedge g = \frac{1}{2}(f + g - |f-g|) \quad \text{and} \quad f \vee g = \frac{1}{2}(f + g + |f-g|)$$

both belong to $\mathscr{L}$ completing the argument. ∎

Theorem 5.2 and 5.3 imply that $\mathscr{L}$ is a vector lattice. Note too that in the proof of 5.3, we have shown that if f is in $\mathscr{L}$, then $|f|$ is also in $\mathscr{L}$.

5.4 THEOREM. If f is in $\mathscr{L}$ and $f \geq 0$ a.e. then $\int f \geq 0$.

PROOF. Let $f = g - h$ a.e. where g and h are in $\tilde{\mathscr{S}}$. Then $f \geq 0$ a.e. implies that $g - h \geq 0$ a.e., thus $g \geq h$ a.e. Hence, by Proposition 4.4

$$\int g \geq \int h$$

Accordingly, $\int f = \int g - \int h \geq 0$. ∎

Using the results of the previous theorems we obtain the next three Corollaries.

5.5. COROLLARY. If f and g are in and $f \leq g$ a.e., then $\int f \leq \int g$.

PROOF. Since $f \leq g$ a.e., then $g - f \geq 0$ a.e.
By Proposition 5.4, we have $\int g - \int f \geq 0$.
Thus

$$\int f \leq \int g \quad .$$

5.6. COROLLARY. <u>If</u> f <u>is</u> <u>in</u> $\mathscr{L}$, <u>then</u> $|f|$ <u>is</u> <u>in</u> $\mathscr{L}$ <u>and</u> $|\int f| \leq \int |f|$.

PROOF. In the Proof of Theorem 5.3, we have shown that if f is in $\mathscr{L}$, then $|f|$ is also in $\mathscr{L}$.

Now, since $f \leq |f|$, then according to Corollary 5.5, $\int f \leq \int |f|$.

Thus

$$|\int f| \leq |\int |f| \, |$$

By Theorem 5.4, since $|f| \geq 0$ implies $\int |f| \geq 0$, we have $|\int f| \leq \int |f|$ as desired.

5.7. COROLLARY. <u>The</u> <u>function</u> f <u>is</u> <u>in</u> $\mathscr{L}$ <u>if</u> <u>and</u> <u>only</u> <u>if</u> f^+ <u>and</u> f^- <u>are</u> <u>in</u> $\mathscr{L}$.

PROOF. Suppose f is in $\mathscr{L}$, then by Corollary 5.6, $|f|$ is in $\mathscr{L}$. In Section 2, we have shown that

$$f^+ = \frac{1}{2}(|f| + f) \qquad \text{and} \qquad f^- = \frac{1}{2}(|f| - f) \quad .$$

Since $\mathscr{L}$ is a vector space, then f^+ and f^- are in $\mathscr{L}$.

Conversely, let f^+ and f^- be in $\mathscr{L}$. In Section 2, we have shown also that $f = f^+ - f^-$.

Thus f is in $\mathscr{L}$.

6. SOME CONVERGENCE THEOREMS

Before proving the basic convergence theorem for functions in $\mathscr{L}$, which is due to the Italian mathematician Beppo Levi (1875-1961), we observe the following very useful result for $\mathscr{L}$.

6.1. PROPOSITION. <u>Given any</u> $\varepsilon > 0$, <u>there exists functions</u> g' <u>and</u> h' <u>in</u> $\tilde{\mathscr{S}}$ <u>such that</u> $f = g' - h'$, $g' \geq 0$ <u>and</u> $h' \geq 0$ a.e., <u>and</u> $\int h' < \varepsilon$.

PROOF. Suppose f is a nonnegative function in $\mathscr{L}$ and $f = g - h$ a.e. where g and h are in $\tilde{\mathscr{S}}$. Since h is in $\tilde{\mathscr{S}}$ there is a nondecreasing sequence $\{h_k\}$ of functions in $\mathscr{S}$ such that $\lim h_k(x) = h(x)$ for a.a.x and $\lim \int h_k = \int h$. Thus, for every $\varepsilon > 0$, there exists an integer k_o such that

$$0 < \int h - \int h_k < \varepsilon \quad \text{whenever} \quad k \geq k_o .$$

Now, let $h' = h + (-h_{k_o})$ and $g' = g + (-h_{k_o})$

Since $-h_{k_o}$ is in $\mathscr{S} \subset \tilde{\mathscr{S}}$, then h' and g' are in $\tilde{\mathscr{S}}$ by Proposition 4.2, and $f = g' - h' = g - h$ a.e. Moreover, $h' \geq 0$ a.e. by the definition of the sequence $\{h_k\}$, and $0 \leq \int h' \leq \varepsilon$.
Also $g' = h' + f$ a.e. implies that $g' \geq 0$ a.e. concluding the proof.

∎

6.2. THEOREM. (LEVI) *Let* $\{f_k\}$ *be a sequence of functions in* $\mathcal{L}$ *which are nonnegative* a.e. *such that* $\{\int \sum_{k=1}^{m} f_k\}$ *is bounded. Then* $\sum_{k=1}^{\infty} f_k(x)$ *converges for* a.a.x *and if*

$f(x) = \sum_{k=1}^{\infty} f_k(x)$ *for* a.a.x *then* f *is in* $\mathcal{L}$ *and*

$$\int f = \sum_{k=1}^{\infty} \int f_k \ .$$

PROOF. For each k, we represent $f_k = g_k - h_k$ a.e. where g_k, h_k are nonnegative functions in $\mathcal{S}$ and $\int h_k \leq \frac{1}{2^k}$ by Proposition 6.1. Since $\{h_k\}$ is a sequence of nonegative functions in $\tilde{\mathcal{S}}$ such that $\int \sum_{k=1}^{m} h_k \leq 1$ for all m, Corollary 4.7 implies that $\sum_{k=1}^{\infty} h_k(x)$ converges for a.a.x, and if

$h(x) = \sum_{k=1}^{\infty} h_k(x)$ for a.a.x, then h is in $\tilde{\mathcal{S}}$ and

$\int h = \sum_{k=1}^{\infty} \int h_k$. Now, since $\{\int \sum_{k=1}^{m} f_k\}$ is bounded, there is a real number c such that $\int \sum_{k=1}^{m} f_k < c$ for all m.

Moreover, let $g_k = f_k + h_k$ so that

$$\int \sum_{k=1}^{m} g_k = \int \sum_{k=1}^{m} f_k + \int \sum_{k=1}^{m} h_k \leq c + 1$$

Therefore, we again appeal to Corollary 4.7 obtaining that $\sum_{k=1}^{\infty} g_k(x)$ converges for a.a.x, and if

$g(x) = \sum_{k=1}^{\infty} g_k(x)$ for a.a.x, then g is in $\tilde{\mathcal{S}}$ and

$\int g = \sum_{k=1}^{\infty} \int g_k$. Since $\sum_{k=1}^{m} f_k = \sum_{k=1}^{m} g_k - \sum_{k=1}^{m} h_k$ it follows that $\sum_{k=1}^{\infty} f_k$ converges for a.a.x and if $f(x) = \sum_{k=1}^{\infty} f_k(x)$, then $f = g - h$ for a.a.x where g and h are in $\tilde{\mathscr{S}}$; thus f is in $\mathscr{L}$.

Finally, $\int f = \int g - \int h = \sum_{k=1}^{\infty} \int g_k - \sum_{k=1}^{\infty} \int h_k$

$$= \sum_{k=1}^{\infty} f_k \quad .$$

∎

We have the corresponding results for sequences.

6.3. COROLLARY. (Monotone Convergence Theorem)

<u>Let</u> $\{g_k\}$ <u>be an</u> a.e. <u>nondecreasing sequence of functions in</u> $\mathscr{L}$ <u>such that</u> $\{\int g_m\}$ <u>is bounded</u>. <u>Then</u> $\{g_k(x)\}$ <u>converges for</u> a.a.x <u>and if</u> $g(x) = \lim g_k(x)$ a.a.x, <u>then</u> g <u>is in</u> $\mathscr{L}$ <u>and</u>

$$\int g = \lim \int g_k \ .$$

PROOF. Let $f_k = g_k - g_{k-1}$ for $k \geq 2$ and let $f_1 = g_1$. Then $\{f_k\}$ is an a.e. nonnegative sequence of functions in $\mathscr{L}$.

Since $\int \sum_{k=1}^{m} f_k = \int (g_{m+1} - g_1) = \int g_{m+1} - \int g_1$

Now g_1 is in $\mathscr{L}$ so let $c = \int g_1$. Then

$$\{\int \sum_{k=1}^{m} f_k\} = \{\int g_{m+1}\} - c \ .$$

Since, $\{\int g_{m+1}\}$ is bounded, it is clear that $\{\int \sum_{k=1}^{m} f_k\}$ is bounded. By Levi's Theorem, $\sum_{k=1}^{\infty} f_k(x)$ converges for a.a.x and since $\sum_{k=1}^{\infty} f_k(x) = \lim_m g_{m+1}(x) - g_1(x)$ for a.a.x Then the limit of $g_m(x)$ exists for a.a.x and $\{g_k(x)\}$ converges for a.a.x.

Also, by Levi's thoerem, if $f(x) = \sum_{k=1}^{\infty} f_k(x)$ for a.a.x, then f is in $\mathcal{L}$.

If $g(x) = \lim g_k(x)$ for a.a.x, then $g(x) = f(x) + g_1(x)$ for a.a.x. Since f and g_1 are in $\mathcal{L}$, then g is in $\mathcal{L}$ and $\int g = \int f + \int g_1$.

Finally, by Levi's Theorem we have

$$\begin{aligned} \int g &= \sum_{k=1}^{\infty} \int f_k + \int g_1 \\ &= \lim \int g_{m+1} - \int g_1 + \int g_1 \\ &= \lim \int g_n \end{aligned}$$

In corollary 6.3 the same conclusion holds if $\{g_k\}$ is an a.e. nonincreasing sequence. The following example shows that the condition that the sequence $\{g_k\}$ be monotone is needed in the previous result.

Example: For any positive integer k let f_k be the function from R into R defined by

$$f_k(x) = \begin{cases} 2^k , & x \in [2^{-k}, 2^{-k+1}] \\ 0 & \text{otherwise} \end{cases}$$

It is clear that $\{f_k\}$ is not a monotone sequence of functions. If $f(x) = \lim_k f_k(x) = \infty$ then $\int f$ is unbounded. On the other hand

$$\int f_k = \int_{2^{-k}}^{2^{-k+1}} 2^k dx = 2^k [x]_{2^{-k}}^{2^{-k+1}} = 1 \quad \text{for all } k$$

So $\lim_k \int f_k = 1$.

Thus $\int f \neq \lim \int f_k$.

From the definition of an integrable function, it is clear that the null set is insignificant in Daniell Integration. More precisely, if f_1 is in $\mathcal{L}$ and $f_1 = f_2$ a.e., then f_2 is in $\mathcal{L}$ and $\int f_1 = \int f_2$.

We next show the converse: if a set is insignificant with respect to integration, then it is a null set.

6.4. PROPOSITION. If f is in $\mathcal{L}$ and $\int |f| = 0$, then $f = 0$ a.e.

PROOF. Suppose f is in $\mathcal{L}$. Let $g_k = k|f|$ for each integer k. Then $\{g_k\}$ is a nondecreasing sequence of fucntions in $\mathcal{L}$ such that $\{\int g_k\}$ is bounded. Therefore, by the Monotone Convergence Theorem, $\{g_k(x)\}$ converges for a.a.x. However,

$g_k(x) = k|f(x)|$ converges only at those points x such that $f(x) = 0$ a.e. (if not so, $g_k(x)$ diverges to ∞) concluding the proof.

The next result characterizes the concept of a null set in terms of the Daniell integral of an appropriate function. We need the following definition.

For any subset E in X, the <u>Characteristic Function</u> χ_E of E is defined as follows:

$$\chi_E = \begin{cases} 1, & \text{for } x \text{ in } E \\ 0, & \text{otherwise} \end{cases}$$

We will establish the following formulas.

(1) $$\chi_{E_1 \cup E_2} = \chi_{E_1} \vee \chi_{E_2}$$

(2) $$\chi_{E_1 \cap E_2} = \chi_{E_1} \wedge \chi_{E_2}$$

(3) $$\chi_{E_1 - E_2} = \chi_{E_1} - (\chi_{E_1} \wedge \chi_{E_2}) .$$

PROOF.

(1) For every x in $E_1 \cup E_2$

$$\chi_{E_1 \cup E_2}(x) = 1 \quad \text{and} \quad \chi_{E_1} \vee \chi_{E_2}(x) = 1 .$$

For every x in $X - (E_1 \cup E_2)$, then

$$\chi_{E_1 \cup E_2}(x) = (\chi_{E_1} \vee \chi_{E_2})(x) = 0$$

(2) For every x in $E_1 \cap E_2$, then

$$\chi_{E_1 \cap E_2}(x) = 1 \quad \text{and} \quad \chi_{E_1} \wedge \chi_{E_2} = 1$$

For every x in $X-(E_1 \cap E_2)$, then

$$\chi_{E_1 \cap E_2}(x) = 0 = (\chi_{E_1} \wedge \chi_{E_2})(x) = 0$$

Hence,

$$\chi_{E_1 \cap E_2} = \chi_{E_1} \wedge \chi_{E_2}$$

(3) For every x in E_1-E_2, implies x is in E_1 but not in E_2, then

$$\chi_{E_1-E_2}(x) = 1$$

and $$(\chi_{E_1} - (\chi_{E_1} \wedge \chi_{E_2}))(x) = 1$$

For every x in $X-(E_1-E_2)$ then x is in E_2 but not in E_1. We have

$$\chi_{E_1-E_2} = 0 \quad \text{and} \quad (\chi_{E_1} - (\chi_{E_1} \wedge \chi_{E_2})(x) = 0 .$$

This concludes the proof.

6.5. COROLLARY. <u>A subset</u> E <u>in</u> X <u>is a null set if and only if</u> χ_E <u>is in</u> $\mathscr{L}$ <u>and</u> $\int \chi_E = 0$.

PROOF. Suppose E is a null set, then $\chi_E = \underline{0}$ a.e. Since $\underline{0}$ is in $\mathscr{L}$, it follows that χ_E is in $\mathscr{L}$ and $\int \chi_E = \int \underline{0} = 0$

Conversely, suppose χ_E is in $\mathscr{L}$ and $\int \chi_E = 0$. Since $\chi_E \geq 0$, we have $\int |\chi_E| = 0$. By Proposition 6.4, $\chi_E = \underline{0}$ a.e. This shows that E is a null set completing the proof.

Before proving another convergence theorem we will prove an important lemma due to the French mathematician Pierre Fatou (1878-1929). First a few basic definitions are helpful.

The <u>lower limit</u> and <u>upper limit</u> of a sequence $\{x_n\}$ are defined, respectively, as follows:

$$\underline{\lim}\, x_n = \lim_{k\to\infty} \inf\{x_n : n \geq k\} = \sup_k \inf\{x_n : n \geq k\}$$

$$\overline{\lim}\, x_n = \lim_{k\to\infty} \sup\{x_n : n \geq k\} = \inf_k \sup\{x_n : n \geq k\}$$

Now let

$$x_k \wedge \ldots \wedge x_j = \inf\{x_k, \ldots, x_j\}$$

$$x_k \vee \ldots \vee x_j = \sup\{x_k, x_{k+1}, \ldots, x_j\}$$

From the above definitions of the lower limit and the upper limit, we can obtain the following characterizations:

$$\underline{\lim}\, x_n = \lim_{k\to\infty} (\lim_{j\to\infty} (x_k \wedge \ldots \wedge x_j))$$

and $$\overline{\lim}\, x_n = \lim_{k\to\infty} (\lim_{j\to\infty} (x_k \vee \ldots \vee x_j))$$

To prove these characteristics, we only have to show that

$$\inf\{x_n : n \geq k\} = \lim_{j\to\infty} (x_k \wedge x_{k+1} \wedge \ldots \wedge x_j)$$

and $$\sup\{x_n : n \geq k\} = \lim_{j\to\infty} (x_k \vee x_{k+1} \vee \ldots \vee x_j) .$$

The proof is completed by definition.

6.6. LEMMA. (Fatou) *If $\{f_k\}$ is a sequence of functions in $\mathscr{L}$ and g is a function in $\mathscr{L}$ such that, for all n, $f_n(x) \geq g(x)$ for a.a.x and $\underline{\lim} \int f_n < \infty$, then $\underline{\lim}\, f_n(x)$ exists for a.a.x and, if $f(x) = \underline{\lim}\, f_n(x)$ for a.a.x, f is in $\mathscr{L}$ and $\int f \leq \underline{\lim} \int f_n$.*

PROOF. Let us consider the lower limit of a sequence of functions in $\mathscr{L}$

$$\underline{\lim}\, f_k(x) = \lim_{k\to\infty} (\lim_{j\to\infty} (f_k(x) \wedge \ \ldots \ \wedge f_j(x)))$$

for all values of x where these limits exist.

Let

$$h_{kj} = f_k \wedge \ldots \wedge f_j \qquad \text{for } j \geq k.$$

For a fixed k, $\{h_{kj}\}$ is a nonincreasing sequence of functions in $\mathcal{L}$ because

$$h_{k,j+1} = h_{kj} \wedge f_{j+1} \leq h_{kj} \quad \text{for all } j,k$$

and $\mathcal{L}$ is a vector lattice. Since for all k, $f_k(x) \geq g(x)$ for a.a.x then $h_{kj} \geq g$ a.e. Therefore, the sequence $\{\int h_{kj}\}$ is bounded below by $\int g$. It follows from the Monotone Convergence Theorem that $\{h_{kj}(x)\}$ converges a.e.

Also, if $h_k(x) = \lim\limits_{j\to\infty} h_{kj}(x)$ for a.a.x, then h_k is in $\mathcal{L}$ and $\int h_k = \lim\limits_{j\to\infty} \int h_{kj}$. Since, for all $j \geq k$, $h_{kj} \leq f_k$ we have $h_k(x) \leq f_k(x)$ for a.a.x and, therefore, $\int h_k \leq \int f_k$. Since $h_{kj} = f_k \wedge h_{k+1,j} \leq h_{k+1,j}$ for all j and k, the sequence $\{h_k\}$ is an a.e. nondecreasing sequence of functions in $\mathcal{L}$. Thus $\{\int h_k\}$ is a nondecreasing sequence of real numbers such that

$$\lim \int h_k = \underline{\lim}_{k} \int h_k \leq \underline{\lim}_{k} \int f_k < \infty$$

Again, by the Monotone Convergence Theorem, $\{h_k(x)\}$ converges for a.a.x and if $h(x) = \lim h_k(x)$ for a.a.x, then h is in $\mathcal{L}$ and

$$\int h = \lim \int h_k \leq \underline{\lim} \int f_k \ .$$

Furthermore, since $h_k(x) = \lim\limits_{j\to\infty} f_k(x) \wedge \quad \dots \wedge f_j(x)$

for all k, then $\underline{\lim}_{k\to\infty} f_k(x) = \underline{\lim}_{k} h_k(x) = \lim h_k(x)$

Therefore, since $\{h_k(x)\}$ converges a.e. then $\underline{\lim}_{k} f_k(x)$ exists a.e., and if $f(x) = \underline{\lim}_{k} f_k(x)$ a.e. then $f(x) = h(x)$ a.e., this shows that f is in $\mathscr{L}$ because h is in $\mathscr{L}$.

Finally, since f = h a.e., we have

$$\int f = \int h = \lim \int h_k \leq \underline{\lim}_{k} \int f_k$$

$\blacksquare$

A similar result holds for the upper limit.

6.7. LEMMA (Fatou) *If* $\{f_k\}$ *is a sequence of functions in* $\mathscr{L}$ *and* g *is a function in* $\mathscr{L}$ *such that, for all* k, $f_k(x) \leq g(x)$ *for* a.a.x *and* $\overline{\lim}_{k} \int f_k > -\infty$, *then* $\overline{\lim}_{k} f_k(x)$ *is finite for* a.a.x *and, if* $f(x) = \overline{\lim}_{k} f_k(x)$ *for* a.a.x, f *is in* $\mathscr{L}$ *and*

$$\int f \geq \overline{\lim}_{k} \int f_k \quad .$$

PROOF. Let $-f_k(x) = h_k(x)$ for all k and all x, and let $-f(x) = h(x)$ for all x. Using Fatou's lemma, the above result is easily established using the fact that

$$\overline{\lim}_{k} f_k(x) = - \underline{\lim}_{k} (-f_k(x))$$

$\blacksquare$

A natural and important question to consider in Integration theory is that of "taking the limit under the Integral sign." Roughly speaking, this means finding conditions under which $\lim \int f_n = \int \lim f_n$. From the two previous lemmas, we obtain the following important Dominated Convergence Theorem.

6.8. THEOREM. (Lebesque Dominated Convergence)

Let $\{f_k\}$ *be a sequence of functions in* $\mathcal{L}$ *such that* $\{f_k(x)\}$ *converges for* a.a.x *and, for all* k, $|f_k| \leq g$ a.e. *for some function* g *in* $\mathcal{L}$. *If* $f(x) = \lim f_k(x)$ *for* a.a.x, *then* f *is in* $\mathcal{L}$ *and*

$$\int f = \lim \int f_k \quad .$$

PROOF. Since for all k, $-g(x) \leq f_k(x) \leq g(x)$ for a.a.x and f_k, g are in $\mathcal{L}$, then

$$-\int g \leq \int f_k \leq \int g$$

and, therefore, $\{\int f_k\}$ is bounded above and below. It follows that

$$\overline{\lim} \int f_k \geq \int (-g) > -\infty \quad \text{and}$$

$$\underline{\lim} \int f_k \leq \int g \quad < \infty$$

Thus, the conditions for Fatou's lemmas 6.6 and 6.7 are satisfied. Therefore, if

$$f(x) = \underline{\lim}\ f_k(x) = \overline{\lim}\ f_k(x) = \lim f_k(x) \text{ a.e.,}$$

then f is in $\mathscr{L}$ and

$$\overline{\lim} \int f_k \leq \int f \leq \lim \int f_k$$

But we always have $\underline{\lim} \int f_k \leq \overline{\lim} \int f_k$.

Hence $\underline{\lim} \int f_k = \overline{\lim} \int f_k = \lim \int f_k = \int f$

completing the proof.

7. MEASURABLE FUNCTIONS

If in the Lebesque Dominated Convergence Theorem we do not have the hypothesis that the sequence $\{f_k\}$ of functions in $\mathscr{L}$ is bounded by an integrable function, then we can no longer conclude that the limit function is integrable.

Example. Consider the following sequence $\{f_k\}$ of functions defined over X = R, where

$$f_k(x) = \begin{cases} 1 & \text{if } x \text{ is in } [-k,k] \\ 0 & \text{elsewhere} \end{cases}$$

In this case, f_k is in $\mathcal{S}$ because for each k, f_k is a step function. Thus, f_k is in $\mathcal{L}$ for each k. Moreover, there does not exist a function g in $\mathcal{L}$ such that $|f_k| \leq g$ a.e. for all k. In fact, $\lim f_k(x) = f(x) = 1$ for every x in R, and we know that $\underline{1}$ is not an integrable function on the whole real line.

Suppose $\{f_k\}$ is a sequence of functions in $\mathcal{L}$ and $\lim f_k(x) = f(x)$ for a.a.x. Take g in $\mathcal{L}$ such that $g \geqslant 0$ and let

$$h_k = (-g) \vee (f_k \wedge g) \text{ . Then}$$

$$h_k(x) = \begin{cases} -g(x) & \text{if} \quad f_k(x) \leq -g(x) \\ f_k(x) & \text{if} \quad -g(x) \leq f_k(x) \leq g(x) \\ g(x). & \text{if} \quad f_k(x) > g(x) \end{cases}$$

h_k is the function f_k cut off above and below by the integrable function g.

Since $\{h_k\}$ is a sequence of functions in $\mathcal{L}$ such that $|h_k| \leq g$ and

$$\begin{aligned} \lim h_k(x) &= \lim_k (-g) \vee (f_k \wedge g)(x) \quad && \text{a.e.} \\ &= (-g) \vee (\lim f_k \wedge g)(x) && \text{a.e.} \\ &= (-g) \vee (f \wedge g)(x) && \text{a.e.} \end{aligned}$$

By the Lebesque Dominated Convergence Theorem, $(-g) \vee (f \wedge g)$ is integrable. This motivates the following definition.

7.1. DEFINITION. A function f from a nomempty set X into R is said to be <u>measurable</u> if, for each g in $\mathcal{L}$ such that $g \geq 0$, $(-g) \vee (f \wedge g)$ is integrable.

We denote the set of measurable functions by $\tilde{\mathcal{M}}$.

We observe that, every entegrable function is measurable, that is $\mathcal{L}$ is a subset of $\tilde{\mathcal{M}}$. For if f is in $\mathcal{L}$, and $g \geq 0$ is integrable, then $(-g) \vee (f \wedge g)$ is integrable (because is a vector lattice).

Now, if f_1 is in $\tilde{\mathcal{M}}$, and $f_1 = f_2$ a.e., then for each g in $\mathcal{L}$ with $g \geq 0$,

$$(-g) \vee (f_1 \wedge g) = (-g) \vee (f_2 \wedge g) \quad \text{for a.e.}$$

Since $(-g) \vee (f_1 \wedge g)$ is integrable, then by previous results in $\mathcal{L}$, $(-g) \vee (f_2 \wedge g)$ is integrable or f_2 is in $\tilde{\mathcal{M}}$. Moreover, it is clear that $\int f_1 = \int f_2$.

If f is in $\tilde{\mathcal{M}}$ and $|f| \leq g$ for g in $\mathcal{L}$ then

$$f = (-g) \vee (f \wedge g) \ .$$

Therefore, f is integrable. That is, <u>a measurable function which is bounded by an integrable function is integrable</u>.

This observation allows us to rewrite the Lebesque Dominated Convergence as follows:

7.2. THEOREM. (Lebesque Dominated Convergence)

If $\{f_k\}$ *is a sequence of measurable functions such that, for all* k, $|f_k| \leq g$ *for some integrable function* g, *and if* $f(x) = \lim f_k(x)$ *for* a.a.x, *then* f *is integrable and* $\int f = \lim \int f_k$.

We now show that unlike $\mathcal{L}$, $\tilde{\mathcal{M}}$ is closed with respect to a.e. pointwise convergence.

7.3. THEOREM. *If* $\{f_n\}$ *is a sequence of functions in* $\tilde{\mathcal{M}}$ *and* $f(x) = \lim f_n(x)$ *for* a.a.x, *then* f *is in* $\tilde{\mathcal{M}}$.

PROOF. Let g be an integrable function such that $g \geq 0$, and let

$$h_k = (-g) \vee (f \wedge g) .$$

Then by definition 7.1, h_k is in $\mathcal{L}$, and $|h_k| \leq g$ because h_k is the function f_k cut off above and below by g. Moreover,

$$\lim h_k(x) = (-g(x)) \vee (f(x) \wedge g(x)) \quad \text{for a.a.x.}$$

Thus, $\{h_k\}$ is as the role of $\{f_k\}$ in the Lebesque Dominated Convergence Theorem. Thus $(-g) \vee (f \wedge g)$ is in $\mathcal{L}$ and, hence, f is in $\tilde{\mathcal{M}}$ by definition 7.1.

∎

In order to show that $\tilde{\mathcal{M}}$ is a vector lattice, the following theorem is useful.

7.4. THEOREM. <u>A function</u> $f:x \to R$ <u>is measurable if and only if the function</u> $g \vee (f \wedge h)$ <u>is integrable for every choice of integrable functions</u> g <u>and</u> h <u>such that</u> $g \leq 0 \leq h$.

PROOF. Let f be a measurable function. Let g and h be the integrable functions such that $g \leq 0 \leq h$. We have two possibilities.

a) If $|g| \leq h$, then we have

$$g \vee (f \wedge h) \leq |g \vee (f \wedge h)| \leq |-h \vee (f \wedge h)|$$

Since $-h \vee (f \wedge h)$ is integrable by definition 7.4, then $g \vee (f \wedge h)$ is integrable.

(b) If $|g| > h$, then we have

$$g \vee (f \wedge h) \leq |g \vee (f \wedge h)| \leq |g \vee (f \wedge (-g))|$$

and by the same argument $g \vee (f \wedge h)$ is integrable. In either case, we always have the same result.

Conversely, if $g \vee (f \wedge h)$ is integrable, we will show that f is a measurable function.

If $|g| \geq h$ then

$$(-h) \vee (f \wedge h) \leq |g \vee (f \wedge h)|$$

Since $g \vee (f \wedge h)$ is integrable, so is $|g \vee (f \wedge h)|$. This implies $(-h) \vee (f \wedge h)$ is integrable. Then by definition f is in $\tilde{\mathcal{M}}$.

If $|g| \leq h$ then

$$g \vee (f \wedge (-g)) \leq |g \vee (f \wedge h)|$$

By the same argument, $g \vee (f \wedge (-g))$ is integrable or f is in $\tilde{\mathcal{M}}$.

This completes the proof.

7.5. PROPOSITION. <u>The class $\tilde{\mathcal{M}}$ forms a vector lattice.</u>

PROOF. First we will show that $\tilde{\mathcal{M}}$ is a real linear space.

Suppose that f_1 and f_2 in $\tilde{\mathcal{M}}$, and $c \neq 0$ in R are all arbitrary. By Theorem 7.4 characterizing measurable functions, if g and h are in $\mathcal{L}$ and $g \leq 0 \leq h$ then

$$g \vee (cf_1 \wedge h) = c\left(\frac{g}{c} \vee \left(f \wedge \frac{h}{c}\right)\right) \quad \text{if} \quad c > 0 \qquad (1)$$

and

$$g \vee (cf_1 \wedge h) = c\left(\frac{h}{c} \vee \left(f_1 \wedge \frac{g}{c}\right)\right) \quad \text{if} \quad c < 0 \qquad (2)$$

It can be verified directly by definition.

Since $\frac{g}{c}$ and $\frac{h}{c}$ are integrable, the function $g \vee (cf_1 \wedge h)$ is integrable or cf_1 is in $\tilde{\mathcal{M}}$. The result is of course trivial if $c=0$.

We now let $p_n = n(h-g)$

and $\quad F_{n1} = (-p_n) \vee (f_1 \wedge p_n)$; and

$$F_{n2} = (-p_n) \vee (f_2 \wedge p_n) \quad .$$

Set $\quad F_n = F_{n1} + F_{n2}$ and observe that, $g \leq 0 \leq h$ implies that $p_n \geq 0$ and p_n is integrable. It then follows that F_{n1} and F_{n2} are integrable by definition; and hence F_n is integrable. Since F_n is integrable, it is measurable by the previous result. Hence $g \quad (F_n \quad h)$ is integrable. Observe further that $g \leq g \quad (-F_n \quad h) \leq h$ where $|g \vee (F_n \wedge h)| \leq (-g) \vee h$.

We now claim that $\{g \vee (F_n \wedge h)\}$ converges to $g \vee ((f_1+f_2) \wedge h)$ as n tends to infinity.

Let $f = f_1+f_2$. If x is such that $h(x) - g(x) = 0$ then $h(x) = g(x) = 0$ (because $g \leq 0 \leq h$) and both of $g \vee (F_n \wedge h)$ and $g \vee (f \wedge h)$ have the value 0 at x; hence this case is disposed of. If x is such that $h(x) - g(x) > 0$, we see that $p_n(x) = n(h-g)(x)$ tends to infinity as n tends to infinity. We have the following three cases:

If $f_1(x)$ is finite, then

$$f_{n1}(x) = (-p_n) \vee (f_1 \wedge p_n)(x) \quad \text{implies} \quad F_{n1}(x) = f_1(x).$$

If $f_1(x) = \infty$, then $F_{n1}(x) = p_n(x)$ and if $f_1(x) = -\infty$, then $F_{n1}(x) = -p_n(x)$. Likewise for F_{n2}. These observations enable us to compute the limit of $F_n(x)$ as follows:

If f_1 and f_2 are finite then $\lim_n F_n(x) = (f_1+f_2)(x)$

If f_1 or f_2 or both are infinite, then the limit of $F_n(x)$ and $(f_1+f_2)(x)$ have the value infinity. Therefore

$$\lim_{n\to\infty} (g \vee (F_n \wedge h)) = g \vee ((f_1+f_2) \wedge h)$$

as claimed. It then follows by Proposition 7.3 that $g \vee ((f_1+f_2) \wedge h)$ is integrable, and hence that cf_1 and f_1+f_2 are in $\tilde{\mathcal{M}}$.

Finally, we will show that $\tilde{\mathcal{M}}$ is a lattice. Suppose f_1 and f_2 are measurable functions. When g and h are in $\mathcal{L}$ and $g \leq 0 \leq h$. Then by (5) and (6) in Section 2, we have

$$\begin{aligned} g \vee ((f_1 \wedge f_2) \wedge h) &= g \vee ((f_1 \wedge h) \wedge (f_2 \wedge h)) \\ &= (g \vee (f_1 \wedge h)) \wedge (g \vee (f_2 \wedge h)) \end{aligned}$$

because is a vector lattice. Thus $f_1 \wedge f_2$ is measurable. Similarly, we have

$$g \vee ((f_1 \vee f_2) \wedge h) = (g \vee (f_1 \wedge h)) \vee (g \vee (f_2 \wedge h))$$

and hence $f_1 \wedge f_2$ is in $\tilde{\mathcal{M}}$. Therefore, $\tilde{\mathcal{M}}$ is a lattice. This concludes the proof.

We observe from the above result that, if f is in $\widetilde{\mathcal{M}}$ then $f^+ = f \vee 0$, $f^- = -(f \wedge 0)$ and $f = f^+ + f^-$ are in $\widetilde{\mathcal{M}}$.

In general, it is not true that $\widetilde{\mathcal{M}}$ is an algebra. However, if the vector lattice $\mathcal{S}$ has the following additional property introduced by the American mathematician Marshall H. Stone (1903-) then $\widetilde{\mathcal{M}}$ is an algebra. We will establish that result in several stages.

The following result is also needed.

7.6. PROPOSITION. <u>If</u> $\{f_k\}$ <u>is a sequence of functions in</u> $\widetilde{\mathcal{M}}$ <u>and</u> $g(x) = \inf_k f_k(x)$, $G(x) = \operatorname{Sup}_k f_k(x)$, $h(x) = \underline{\lim}_k f_k(x)$ <u>and</u> $H(x) = \overline{\lim}_k f_k(x)$ <u>for</u> a.a.x, <u>then</u> g, G, h <u>and</u> H <u>are in</u> $\widetilde{\mathcal{M}}$

PROOF. Let $g_k = \bigwedge_{j=1}^{k} f_j$, $G_k = \bigvee_{j=1}^{k} f_j$, then, by definition and results concerning $\underline{\lim}$ and $\overline{\lim}$, $g(x) = \lim g_k(x)$ and $G(x) = \lim G_k(x)$ for a.a.x. Because $\widetilde{\mathcal{M}}$ is a vector lattice, $\{g_k\}$ and $\{G_k\}$ are sequences of functions in $\widetilde{\mathcal{M}}$. Therefore, g and G are in $\widetilde{\mathcal{M}}$ by 7.3.

Next, let $h = \lim f'_k$ and $H = \lim f''_k$

where $$f'_k = \lim_{j\to\infty} (f_k \wedge \ \dots \ \wedge f_j)$$

and $$f''_k = \lim_{j\to\infty} (f_k \vee \ \dots \ \vee f_j)$$

Propositions 7.3 and 7.5 imply that h and H are in $\widetilde{\mathcal{M}}$.

∎

7.7. STONE's AXIOM. *If f is in $\mathcal{S}$, then $\underline{1} \wedge f$ is also in $\mathcal{S}$.*

7.8. PROPOSITION. *If $\mathcal{S}$ satisfies Stone's Axiom, then $\underline{1}$ is in $\tilde{\mathcal{M}}$.*

PROOF. First we prove that $\tilde{\mathcal{S}}$ also satisfies Stone's Axiom. Let h be in $\tilde{\mathcal{S}}$, then there exists a nondecreasing sequence $\{h_k\}$ of functions in $\mathcal{S}$ such that $\lim h_k(x) = h(x)$ for a.a.x and $\{\int h_k\}$ is bounded. Therefore, $\{\underline{1} \wedge h_k\}$ is a nondecreasing sequence of functions in $\mathcal{S}$ (by assumption) and $\lim_k(\underline{1} \wedge h_k) = \underline{1} \wedge h$. Also, $\underline{1} \wedge h_k \leq h_k$ for all k so that $\int (\underline{1} \wedge h_k) \leq \int h_k$. Hence, $\{ \int (\underline{1} \wedge h_k)\}$ is bounded. Thus, $(\underline{1} \wedge h)$ is in $\tilde{\mathcal{S}}$ as claimed.

Now let f in $\mathcal{L}$, with $f \geq 0$, be arbitrary. We want to show that $(-f) \vee (1 \wedge f)$ is in $\mathcal{L}$. Since f is in $\mathcal{L}$, there exists g and h in $\tilde{\mathcal{S}}$ such that $f = g - h$ a.e., and $g \geqslant h$. Moreover, since g and h are in $\tilde{\mathcal{S}}$, then there are two nondecreasing sequences $\{g_k\}$ and $\{h_k\}$ in $\mathcal{S}$ with $\lim g_k(x) = g(x)$ and $\lim h_k(x) = h(x)$ for a.a.x and $\{\int g_k\}$ and $\{\int h_k\}$ both bounded, say by G and H for definiteness. Fix i and consider the sequence $\{g_k - g_i\}$: it is a nondecreasing sequence of functions in $\mathcal{S}$ such that

$\int (g_k-g_i) \leq \int g_k + \int h_i \leq G+H$ so that the sequence $\{ \int (g_k-h_i)\}$ is bounded. Thus

$$\lim_k (g_k-h_i)(x) = \lim g_k(x) - h_i(x) = g(x) - h_i(x) \text{ a.e.}$$

so that by the definition of $\tilde{\mathscr{S}}$, $g-h_i$ belongs to $\tilde{\mathscr{S}}$ for every positive integer i.

We define $f_k = g-h_k$ so that f_k belongs to $\tilde{\mathscr{S}}$ for every k. Since $\{h_k\}$ is a nondecreasing sequence, it follows that $\{f_k\}$ is a nonincreasing sequence of functions in $\tilde{\mathscr{S}}$. Since $f_k \wedge \underline{1}$ belongs to $\tilde{\mathscr{S}}$ we have $\{f_k \wedge 1\}$ is a nonincreasing sequence in $\tilde{\mathscr{S}}$. Moreover,

$$\lim f_k(x) = g(x) - \lim h_k(x) = g(x) - h(x) \quad \text{a.e.}$$
$$= f(x) \quad \text{for a.a.x}$$

Thus, $\lim(f_k(x) \wedge 1) = f(x) \wedge 1$ for a.a.x.

Also, by the first part, since g-h is in $\tilde{\mathscr{S}}$, then $(g-h) \wedge \underline{1}$ is also in $\tilde{\mathscr{S}}$ and from $(g-h) \wedge \underline{1} \leqslant (g-h_k) \wedge \underline{1}$ for every k, we obtain

$$\int ((g-h) \wedge \underline{1}) \leq \int (f_k \wedge \underline{1}) \quad \text{for all k.}$$

Thus, $\{\int(f_k \wedge \underline{1})\}$ is a sequence in $\tilde{\mathscr{S}} \subset \mathscr{L}$ bounded below. By the Monotone Convergence Theorem 6.3 we have $\lim(f_k \wedge \underline{1})(x) = (f \wedge \underline{1})(x)$ for a.a.x. This implies that $f \wedge \underline{1}$ belongs to $\mathscr{L}$.

Finally, (-f) is in $\mathscr{L}$ so that $(-f) \vee (f \wedge \underline{1})$ belongs to $\mathscr{L}$ since $\mathscr{L}$ is a vector lattice. Thus $\underline{1}$ belongs to $\mathscr{M}$.

We can use these results to show that $\tilde{\mathcal{M}}$ is an algebra.

7.9. PROPOSITION. <u>If $\mathcal{S}$ satisfies Stone's Axiom then $\tilde{\mathcal{M}}$ is an algebra.</u>

PROOF. Since $\mathcal{S}$ satisfies Stone's Axiom, by 7.7, $\underline{1}$ is in $\tilde{\mathcal{M}}$. Since $\tilde{\mathcal{M}}$ is a vector lattice, then for every real number r, we define $\underline{r} = r\cdot\underline{1}$, therefore $\underline{r}$ belongs to $\tilde{\mathcal{M}}$.

We now show that if f is in $\tilde{\mathcal{M}}$, then f^2 is also in $\tilde{\mathcal{M}}$.

Let $\{r_k : k=1,2,\dots\}$ be an enumeration of the rational numbers in R. For each x in X, $\inf_k (f(x)-r_k)^2 = 0$ so that

$$f^2(x) + \inf_k(-2r_k f(x) + r_k^2) = 0$$

Therefore, $f^2(x) = -\inf_k(-2r_k f(x) + r_k^2)$

$$= \sup_k (2r_k f(x) - r_k^2) \quad .$$

So for each k, $2r_k f(x) - r_k^2$ is in $\tilde{\mathcal{M}}$ (because $\tilde{\mathcal{M}}$ is a vector lattice), so by Proposition 7.6, f^2 is in $\tilde{\mathcal{M}}$. Now for any f,g in $\tilde{\mathcal{M}}$, we have

$$f\cdot g = \frac{1}{4}\,((f+g)^2 - (f-g)^2) \quad .$$

Thus, $f\cdot g$ are in $\tilde{\mathcal{M}}$ which shows that $\tilde{\mathcal{M}}$ is an algebra.

We know that if f is measurable and if $|f|$ is integrable, then $f = (-|f|) \vee (f \wedge |f|)$ implies that f is integrable.

8. MEASURABLE SETS

8.1. DEFINITION. A subset E of X is <u>measurable</u> if χ_E is a measurable function; E is <u>integrable</u> if χ_E is integrable.

We will denote the set of measurable subsets of X by $\mathcal{M}$.

Note that if a set is integrable, then it is measurable. If $\mathcal{S}$ satisfies Stone's Axiom, although the function $\underline{1}$ is measurable even though it is not integrable by 7.7. Hence the whole space X is a measurable set (because $\chi_X = \underline{1}$ is in $\tilde{\mathcal{M}}$).

A collection $\mathcal{D}$ of sets in the power $\mathcal{P}(X)$ is called an <u>algebra of sets</u> if X is in $\mathcal{D}$ and if $A \cup B$ and A-B are in $\mathcal{D}$ whenever A and B are in $\mathcal{D}$. An algebra $\mathcal{D}$ is called a <u>σ-algebra</u> if any countable union of sets in $\mathcal{D}$ is in $\mathcal{D}$.

Define <u>ring of sets</u> and <u>σ-ring</u>.

A collection $\mathcal{B}$ of sets in $\mathcal{P}(X)$ is called a <u>ring of sets</u> if $A \cup B$ and $A - B$ are in $\mathcal{B}$ whenever A, B are in $\mathcal{B}$. It is a <u>σ-ring</u> if it is a ring that is closed under the formulation of countable unions.

Thus an algebra of sets is simply a ring of sets that contains the whole set X.

8.2. PROPOSITION. <u>The collection $\mathcal{M}$ of measurable sets is a σ-ring.</u>

PROOF. We see that the empty set $\emptyset$ is in $\mathcal{M}$ since the characterstic function $\chi_\emptyset = 0$ is integrable and therefore measurable.

Next we show that $\mathcal{M}$ is a ring of sets.

Since $\tilde{\mathcal{M}}$ is a vector lattice, then $\chi_{E_1} \vee \chi_{E_2}$, $\chi_{E_1} \wedge \chi_{E_2}$ and $\chi_{E_1} - (\chi_{E_1} \wedge \chi_{E_2})$ are measurable functions. Therefore, $\chi_{E_1 \cup E_2}$, $\chi_{E_1 - E_2}$ are measurable functions by the previous results of Characteristic function. Thus, $E_1 \cup E_2$ and $E_1 - E_2$ are measurable sets. It is then clear that $\mathcal{M}$ is a ring. We will show that $\mathcal{M}$ is a σ-ring. Suppose $E_1, E_2, \ldots$ are in $\mathcal{M}$. Let $G_n = \bigcup_{i=1}^{n} E_i$ and $G = \bigcup_{i=1}^{\infty} E_i$, also let $g_n = \chi_{G_n} = \bigvee_{i=1}^{n} \chi_{E_i}$ and $g = \chi_G$. It is easy to see that $\{g_n\}$ is an increasing sequence of measurable functions that converges to g. For all n, G_n is in $\mathcal{M}$, because $\mathcal{M}$ is a ring, and therefore g_n is in $\tilde{\mathcal{M}}$. By proposition 7.3, g is a measurable function, where G is in $\mathcal{M}$. Thus, $\mathcal{M}$ is a σ-ring.

8.3 PROPOSITION. <u>If $\mathcal{S}$ satisfies Stone's Axiom, then $\mathcal{M}$ is a σ-algebra.</u>

PROOF. By 8.2, $\mathcal{M}$ is a -ring. In the first part of our discussion in Section 8, we showed that X is a measurable set whenever $\mathcal{S}$ satisfies Stone's Axiom. Now $\mathcal{M}$ is a σ-algebra whenever $\mathcal{S}$ satisfies Stone's Axiom.

9. MEASURE

The theory of measure began to develop in a period when the attention of mathematicians was being concentrated on the importance of examining very general kinds of point sets in Euclidean space. It was necessary to seek, for these point sets on the real line, something which specialized to length when the point-set was an interval; likewise, for point-sets in the plane, the "measure" was area. The classical theory of measure and integration, developed by Lebesque, stayed mainly with the sets in Euclidean space and with real functions defined on Euclidean space. Now, the development of measure theory and its applications can be relieved from the limitations of Euclidean space.

In this section we will define a measure for measurable sets which is a generalization of the volume of an interval. The volume of an interval (a,b) is $\int \chi_{(a,b)}$.

The following rules hold for these operations of addition and subtraction with the extended ideal number ∞: if r is a real number, then $\infty \pm r = \infty$, and $\infty + \infty = \infty$. The operation $\infty - \infty$ is not defined and is avoided.

9.1. DEFINITION. A <u>measure</u> is a nonnegative extended real-valued function defined on a σ-ring $\mathcal{M}$ of measurable sets as follows: If E is in $\mathcal{M}$, then

$$m(E) = \begin{cases} \int \chi_E & \text{if } E \text{ is integrable} \\ \infty & \text{, otherwise} \end{cases}$$

Recall that, we have shown that E is a null set if and only if $\int \chi_E = 0$ (Corollary 6.5).

9.2 THEOREM. <u>If</u> E <u>and</u> F <u>are measurable sets, we obtain the following properties for the measure</u> m:

(9.3) $m(\emptyset) = 0$

(9.4) If $E \subset F$ implies $m(E) \leq m(F)$

(9.5) $m(E \cup F) \leq m(E) + m(F)$

(9.6) If $E \cap F = \emptyset$ implies $m(E \cup F) = m(E) + m(F)$

(9.7) If $E \subset F$ and $m(E) < \infty$ implies
$m(F-E) = m(F) - m(E)$

PROOF.

Proof of (9.3): By the result of Corollary 6.5 because $\emptyset$ is a null set.

Proof of (9.4): We have two cases.

If F is not integrable, then $m(F) = \infty$, then $m(E) \leq m(F) = \infty$.

If F is integrable, then χ_F is integrable. Since $E \subset F$, then $\chi_E \leq \chi_F$, therefore, χ_E is integrable and

$$m(E) = \int \chi_E \leq \int \chi_F = m(F) .$$

Proof of (9.5): There are two cases.

If E is not integrable, then $m(E) = \infty$. Since $E \subset E \cup F$, $E \cup F$ is not integrable and $m(E \cup F) = \infty$. Thus, the result holds. The same is true if F is not integrable.

If E and F are both integrable, then by Proposition 8.3, $\chi_{E \cup F} = \chi_E \vee \chi_F \leq \chi_E + \chi_F$. Therefore,

$m(E \cup F) = \int \chi_{E \cup F} \leq \int \chi_E + \int \chi_F = m(E) + m(F)$.

Thus, using induction it is easy to show that

$$m(\bigcup_{k=1}^{n} E_i) \leq \sum_{i=1}^{n} m(E_i) \quad \text{for every positive integer } n.$$

Proof of (9.6): If $E \cap F = \emptyset$, then it is easy to show that $\chi_{E \cup F} = \chi_F + \chi_F$.

Now, suppose that E is not integrable. Then $E \cup F$ is not integrable and, therefore, $m(E) = \infty$ and $m(E \cup F) = \infty$. Similarly if F is not integrable. Thus the result holds in these cases. On the other hand, if E and F are both integrable, then $\chi_{E \cup F} = \chi_E + \chi_F$ implies that

$$m(E \cup F) = \int \chi_{E \cup F} = \int \chi_E + \int \chi_F = m(E) + m(F).$$

In either case we obtain 9.6.

If $E_1, E_2, \ldots, E_n$ are in $\mathcal{M}$, and are pairwise disjoint sets in $\mathcal{M}$, then using induction, it is easy to show that

$$m(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} m(E_i)$$

for every positive integer n. From this result, we can say that m is finitely additive.

Proof of 9.7: If F is not integrable, then $m(F) = \infty$. Since $E \subset F$ and $(F-E) \cap E = \emptyset$, it follows from 9.6 and $F = (F-E) \cup F$ that

$$m(F) = m((F-E) \cup E) = m(F-E) + m(E) .$$

Therefore $m(F-E) = m(F) - m(E)$ as desired.

Let $E_1, E_2, \ldots$ be measurable sets and let the E_i's be pairwise disjoint. If

$$\lim_{n\to\infty} \sum_{i=1}^{n} m(E_i)$$

exists and

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i)$$

we say that m is <u>countably additive</u>.

A basic property of a measure is that it is countably additive. We now prove the following theorem.

9.8. <u>THEOREM</u>. <u>The measure</u> m <u>is countably additive</u>.

PROOF. Let $\{E_k\}$ be a sequence of pairwise disjoing measurable sets and let $E = \bigcup_{k=1}^{\infty} E_k$. We will show that E is measurable.

Let $F_n = \bigcup_{k=1}^{n} E_k$ then by property 9.6

$$\chi_{F_n} = \chi_{\bigcup_{k=1}^{n} E_k}$$

and $\quad m(F_n) = \sum_{k=1}^{n} m(E_k)$ and χ_{F_n} is measurable. By

7.3 we have

$$\chi_E = \lim_{n\to\infty} \chi_{F_n} = \sum_{k=1}^{\infty} \chi_{E_k} \qquad \text{for a.a.x.}$$

The limit χ_E is also a measurable function. Thus, E is measurable as claimed.

Now, if for some k, E_k is not integrable so that $m(E_k) = \infty$, then $\sum_{k=1}^{\infty} m(E_k) = \infty$. Also, since $E_k \subset E$, then $m(E) = \infty$, and therefore, $m(E) = \sum_{k=1}^{\infty} m(E_k)$. On the other hand, suppose that for all k, E_k is integrable, then $\sum_{k=1}^{\infty} m(E_k)$ either converges or else it diverges to infinity.

Assume $\sum_{k=1}^{\infty} m(E_k) = \infty$.

For all n, $\bigcup_{k=1}^{n} E_k \subset E$, then by 9.7

$$m(E) \geq m\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m(E_k) .$$

This inequality is true for all n, so that

$$\infty \geq m(E) \geq \sum_{k=1}^{\infty} m(E_k) = \infty .$$

Finally, suppose $\sum_{k=1}^{\infty} m(E_k)$ converges.

Then $\sum_{k=1}^{n} m(E_k) \leq \sum_{k=1}^{\infty} m(E_k)$ from which it follows that

$\{\int \sum_{k=1}^{n} m(E_k)\}$ is bounded. Since $\chi_E = \sum_{k=1}^{\infty} \chi_{E_k}$, by Levi's

theorem 6.2, $\sum_{k=1}^{\infty} \chi_{E_k}$ converges and $\chi_E = \sum_{k=1}^{\infty} \chi_{E_k}$ is

integrable. Also E is integrable and

$$m(E) = \int \chi_E = \sum_{k=1}^{\infty} \int \chi_{E_k} = \sum_{k=1}^{\infty} m(E_k)$$

concluding the proof of the theorem.

∎

If we drop the requirement that the sets be pairwise disjoint in the previous result, we obtain the following proposition whose proof is similar to that of Theorem 9.8 above.

9.9. PROPOSITION. <u>If</u> $\{E_k\}$ <u>is a sequence of measurable sets, then</u>

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m(E_k) \quad .$$

In the next chapter, we will consider the concept of measure and its theoretical aspects in greater detail.

III. SECOND APPROACH TO INTEGRATION VIA MEASURE THEORY

1. INTRODUCTION

Measure theory in recent years has shown itself to be most useful for its applications in modern analysis. In this chapter, the approach to integration is via measure theory.

The development begins with a nonempty set X, a σ-ring $\mathcal{M}$ of subsets of X, and a countably additive, nonnegative, extended real-valued function m defined on $\mathcal{M}$ with the property that $m(\emptyset) = 0$. Then $(X, \mathcal{M}, m)$ is called a <u>measure space</u>. The subsets of X which are in $\mathcal{M}$ are called measurable sets and m is called a measure. The value $m(E)$ is called the measure of E.

The measure m may be induced by an "outer measure" by reducing the domain of m* to the available σ-ring $\mathcal{M}$ of measurable sets.

After we obtain a measure m on the σ-ring $\mathcal{M}$ of measurable subsets of X, and by a parallel formulation that characterizes the condition of a continuous function, we define the concept of a "measurable function" $f: X \to R$. The class of measurable functions forms a vector lattice as well as an algebra.

We next define the class of integrable functions on a measurable set E in three steps. First, the integral of a simple function is defined which is then extended to the class of integrable nonnegative measurable functions. (This is done

by taking sequences of nonnegative simple functions, each of which is integrable function on E.) Finally, utilizing the relation $f = f^+ - f^-$ we can obtain the integral of any measurable function which may also assume negative values.

The chapter concludes with the very important Lebesque Dominated Convergence Theorem which gives conditions under which $\lim \int_E f_n = \int_E \lim f_n$; i.e., under which the operations of integration and taking limits of sequences can be interchanged.

2. ADDITIVE SET FUNCTIONS

We now consider a nonnegative extended real-valued function $m: \mathcal{M} \to R$, where $\mathcal{M}$ is a nonempty class of subsets of a nonempty set X. Such a function is called <u>additive</u> if the range of m does not contain both $-\infty$ and $+\infty$ and if

$$(2.1) \qquad m(E_1 \cup E_2) = m(E_1) + m(E_2) \; ;$$

whenever E_1 and E_2 are in $\mathcal{M}$ such that $E_1 \cup E_2$ is in $\mathcal{M}$ and $E_1 \cap E_2 = \emptyset$.

In the case that $\mathcal{M}$ is a ring, we can prove by induction that the additive function m is <u>finitely additive</u> in the following sense: if $E_1, E_2, \ldots, E_n$ are disjoint members of $\mathcal{M}$, then

$$(2.2) \qquad m\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m(E_i) \quad .$$

From this, if there exists E in $\mathcal{M}$ such that $m(E)$ is finite, then $E \cup \emptyset = E$ and $E \cap \emptyset = \emptyset$ implies $m(E \cup \emptyset) = m(E) + m(\emptyset) = m(E)$. Hence $m(\emptyset) = 0$.

If m is additive on the ring $\mathcal{M}$, and if E, F are in $\mathcal{M}$, with $F \subset E$ and $m(F)$ finite, it is easily shown that

$$m(E - F) = m(E) - m(F) \tag{2.3}$$

The above is called the <u>subtractive property</u> of m. To prove (2.3), we write $E = (E-F) \cup F$, use the fact that $m(F)$ is finite, and apply (2.1).

Let $\mathcal{M}$ be a ring. If the measure m is additive and has nonnegative values, then it is <u>monotone</u>: that is $m(E) \leq m(F)$ whenever E,F in $\mathcal{M}$ and $E \subset F$. For if E and F are as indicated, we can write

$$F = E \cup (F-E) \quad \text{and} \quad m(F) = m(E) + m(F-E) .$$

Since $m(F-E) \geq 0$, it follows that $m(E) \leq m(F)$.

Now, suppose the range of $m: \mathcal{M} \to R^* = R \quad \{-\infty, +\infty\}$ has at most one of the values $-\infty$, $+\infty$. We recall that m is <u>countably additive</u> if, whenever $E_1, E_2, \ldots$ and $\bigcup_i E_i$ are in $\mathcal{M}$ and the E_i's are pairwise disjoint, then

$$\lim_{n\to\infty} \sum_{i=1}^{n} m(E_i)$$

exists in R*, and

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i).$$

From the concept of a countably additive set function, we next define a measure on $\mathcal{M}$ and consider its properties.

We will see later that the notion of a Borel set plays an important role in measure theory. These sets are defined as follows:

The class $\mathcal{B}$ of <u>Borel Sets</u> is the smallest σ-algebra which contains all of the open sets. Equivalently, it is the smallest σ-algebra which contains all of the closed sets.

3. MEASURES — PROPERTIES OF MEASURES

We shall be mainly interested in the case where $m(E) \geq 0$ for every E in $\mathcal{M}$. If $m: \mathcal{M} \to R^*$ is countably additive, has nonnegative values and if $m(\emptyset) = 0$, we call m a <u>measure</u> on $\mathcal{M}$.

The value m(E) is called the <u>measure</u> of E. Usually $\mathcal{M}$ will be a σ-ring of sets.

Let m be a measure on a ring $\mathcal{M}$. If m(E) is finite for each E in $\mathcal{M}$, we say that m is a <u>finite measure</u>.

A measure m on a ring $\mathcal{M}$ is called σ-finite if each E in $\mathcal{M}$ is contained in some countable union, $E \subset \bigcup_i E_i$, where E_i is in $\mathcal{M}$ and $m(E_n)$ is finite.

Since $E \cap E_i$ in $\mathcal{M}$ and $m(E \cap E_i)$ is also finite, it follows from $E = \bigcup_i (E \cap E_i)$ that m is <u>σ-finite if and only if</u>

each element of $\mathcal{M}$ *is expressible as a countable union of members of* $\mathcal{M}$*; each having finite measure.*

Sequences of sets play a vital role in measure theory, and for that reason we will need some definitions and some basic properties of sequences of sets.

Let $\{E_n\}$ be a sequence of sets. We define

$$\overline{\lim_n}\, E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n \quad .$$

$$\underline{\lim_n}\, E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} E_n$$

These sets are called the *limit superior* and *limit inferior*, respectively, of the sequence of sets.

Thus, $\overline{\lim_n}\, E_n$ consists of those points which belong to E_n for an infinite number of values of n. A similar analysis shows that $\underline{\lim_n}\, E_n$ consists of those points which belong to E_n for all except a finite number of values of n. It is clear then that for any sequence $\{E_n\}$ of sets, we always have $\underline{\lim_n}\, E_n \subset \overline{\lim_n}\, E_n$.

A sequence $\{E_n\}$ of sets is said to be *convergent* if

$$\underline{\lim_n}\, E_n = \overline{\lim_n}\, E_n = \lim E_n \quad .$$

A sequence $\{E_n\}$ of sets is called a *nondecreasing sequence* if for each positive integer n,

$$E_n \subset E_{n+1} \quad .$$

It is called a <u>nonincreasing sequence</u> if for each n,

$$E_n \supset E_{n+1} \quad .$$

A <u>monotone sequence</u> is one which is either a nondecreasing sequence or a nonincreasing sequence.

We have a very useful result: every monotone sequence is convergent.

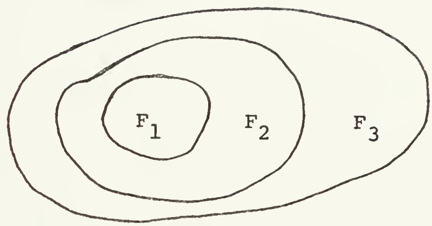
To prove this, suppose $\{E_n\}$ is a nondecreasing sequence of sets; then for each i, $E_i \quad \bigcap_{n=i}^{\infty} E_n$; therefore $\underline{\lim}_n E_n = \bigcup_{i=1}^{\infty} E_i$. However, for a nondecreasing sequence $\bigcup_{n=i}^{\infty} E_n$ is independent of i, so we make take i=1 and

$$\overline{\lim_n}\, E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} E_n = \underline{\lim}_n\, E_n \, .$$

The nonincreasing sequence set may be treated similarly. In that case, $\lim E_n = \bigcap_{n=1}^{\infty} E_n$.

The following observation indicates a useful way of representing unions of sets as a union of disjoint sets. Suppose that certain sets $E_1, E_2, \ldots$ are given (the number may be finite or countably infinite). Define new sets F_1, F_2 inductively as follows:

$$(3.1) \qquad \begin{cases} F_1 = E_1 \\ F_{i+1} = E_{i+1} - (E_1 \cup \ldots \cup E_i) & \text{if } i \geq 1 \end{cases}$$



Then $F_n \subset E_n$ for each n, and $\{F_i\}$ is a sequence of disjoint sets, i.e. $F_i \cap F_j = \emptyset$ if $i \neq j$, and

$$(3.2) \qquad \bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i$$

If we write $G_n = \bigcup_{i=1}^{n} E_i$, then

$$G_1 = E_1 \quad , \quad F_{i+1} = G_{i+1} - G_i \qquad \text{if } i \geq 1$$

and (3.2) can be written as

$$(3.3) \qquad \bigcup_{i=1}^{n} F_i = G_n$$

The situation is shown schematically in the above figure, when G_n is represented as being composed of a number of disjoint sets $F_1, F_2, \ldots, F_n$.

Throughout this section let $\mathcal{M}$ be a ring of sets and let $m: \mathcal{M} \to R^*$ be a measure. We will consider some useful theorems for which we will frequently find applications.

3.4. THEOREM. Suppose $E, E_1, E_2, \ldots$ are elements of $\mathcal{M}$ such that $E \subset \bigcup_i E_i$, then

$$m(E) \leq \sum_i m(E_i) \quad .$$

PROOF. Let $F_i = E \cap E_i$. Define $G_1 = F_1$ and by induction define $G_i = F_i - (\bigcup_{j=1}^{i-1} F_j)$, if $i \geq 2$.

Then $G_i \subset F_i \subset E_i$ so that $m(G_i) \leq m(E_i)$ for all i. Also,

$$E = E \cap (\bigcup_i E_i) = \bigcup_i (E \cap E_i) = \bigcup_i F_i = \bigcup_i G_i$$

Now, we use the observation described in (3.1) since m is countably additive, we have

$$m(E) = \sum_i m(G_i) \leq \sum_i m(E_i)$$

∎

Conversely, we have the following theorem

3.5. THEOREM. Let $\{E_i\}$ be a disjoint countable collection of elements of $\mathcal{M}$, and let E be any member in $\mathcal{M}$ such that $\bigcup_i E_i \subset E$. Then

$$\sum_i m(E_i) \leq m(E).$$

PROOF. The result is clear in the case that the number of E_i's is finite. In the infinite case, by hypothesis, since $\bigcup_i E_i \subset E$ belongs to $\mathcal{M}$, and since m is countably additive, we have

$$m(\bigcup_i E_i) = \sum_i m(E_i)$$

Thus, since m is monotone, from $\bigcup_i E_i \subset E$ we have

$$\sum_i m(E_i) \leq m(E) .$$

∎

From the convergence of the monotone sequences of sets, we have the limits of the measure of sets as follows.

3.6. THEOREM. Let $\{E_n\}$ be a nondecreasing sequence of members of a ring $\mathcal{M}$ and $\bigcup_{n=1}^{\infty} E_n$ belongs to $\mathcal{M}$, then

$$m(\bigcup_{j=1}^{\infty} E_n) = \lim_n m(E_n) .$$

PROOF. Let $F_1 = E_1$, $F_n = E_n - E_{n-1}$. Then the F_n's are disjoint and $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n$. Hence

$$m(\bigcup_{n=1}^{\infty} E_n) = m(\bigcup_{n=1}^{\infty} F_n) = \sum_{n=1}^{\infty} m(F_n) = \lim_{n\to\infty} \sum_{k=1}^{n} m(F_k)$$

But $\sum_{k=1}^{n} m(F_k) = m(\bigcup_{k=1}^{n} F_k) = m(\bigcup_{k=1}^{n} E_k) = m(E_n)$

because $\bigcup_{k=1}^{n} E_k$ for all $n \geq 1$

Thus $m(\bigcup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \sum_{k=1}^{n} m(F_k) = \lim_n m(E_n)$.

∎

3.7. THEOREM. Let $\{E_n\}$ be a nonincreasing sequence of members of the ring $\mathcal{M}$ such that $\bigcap_{n=1}^{\infty} E_n$ belongs to . If $m(E_1) < \infty$, then

$$m(\bigcap_{n=1}^{\infty} E_n) = \lim_n m(E_n) .$$

PROOF. Since $\{E_n\}$ is a nonincreasing sequence of measurable sets, then $E_n \subset E_1$ for all n. Also m is monotone so that $m(E_n) \leqslant m(E_1) < \infty$ for every $n > 1$. Hence $\bigcap_{n=1}^{\infty} E_n \subset E_1$ implies that $m(\bigcap_{n=1}^{\infty} E_n)$ is finite.

Let $F_n = E_1 - E_n$, $E = \bigcap_{n=1}^{\infty} E_n$. Then $F_n \subset F_{n+1}$ and, by DeMorgan's law, we have

$$E_1 - E = E_1 - (\bigcap_{n=1}^{\infty} E_n) = \bigcup_{n=1}^{\infty} (E_1 - E_n) = \bigcup_{n=1}^{\infty} F_n .$$

Therefore, by 3.6,

$$m(E_1 - E) = m(\bigcup_{n=1}^{\infty} F_n) = \lim_n m(F_n) = \lim_n m(E_1 - E_n) .$$

But $E_1 = E \cup (E_1 - E)$, so $m(E_1) = m(E) + m(E_1 - E)$.

Since all three values of m here are finite, we have

$$m(E_1 - E) = m(E_1) - m(E)$$

In the same way, $m(E_1 - E_n) = m(E_1) - m(E_n)$.

Thus, $m(E_1 - E) = m(E_1) - m(E) = m(E_1) - \lim_n m(E_n)$

This implies

$$m(E) = \lim_n m(E_n) .$$

In the next section, we will define an outer measure function m* which is not, in general, countably additive. From that outer measure, however, we will obtain a measure.

4. THE MEASURE INDUCED BY AN OUTER MEASURE

The construction of a measure proceeds in the following way: First, we construct a function called an <u>outer measure</u>. This function, which we shall denote by m*, has for its domain the class of <u>all</u> subsets of X, and its range lies in R*, the extended real number system. The values of m* are nonnegative, m* is monotone, and m* has a property like additivity. But m* is not countably additive, and hence it is not a measure. Thus we will diminish the domain of m* in such a way that m* actually becomes a measure on the reduced domain. This

successful attempt leads to the σ-ring of the class of measurable sets. If E is a measurable set, then $m^*(E)$ will be called the <u>measure of</u> E.

4.1. DEFINITION. Let X be an arbitrary nonempty set, and m^* be a function whose domain is the family of all subsets of X and whose range lies in R^* such that m^* satisfies the following conditions:

1) $m^*(\emptyset) = 0$
2) $m^*(E) \geq 0$ for each E in X.
3) If $E \subset F$, then $m^*(E) \leq m^*(F)$.
4) If $\{E_n\}$ is any countable collection of subsets of X, then $m^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} m^*(E_n)$.

Under these conditions, we call m^* an <u>outer measure on the σ-ring $\mathcal{M}$ consisting of all subsets of</u> X.

If X is a metric space and if m^* is an outer measure on $\mathcal{M}$, which satisfies the following additional condition:

5) $m^*(E \cup F) = m^*(E) + m^*(F)$, whenever E and F are nonempty sets which have a positive distance apart, then we say that m^* is a <u>metric outer space</u> on $\mathcal{M}$.

Since m^* is not countably additive and hence it is not a measure. We now turn to the general theory of obtaining an outer measure on $\mathcal{M}$. The problem of restricting the domain of m^* in such a way as to obtain a measure on $\mathcal{M}$ is solved on the basis of the following definition:

4.2. DEFINITION. A set E in X is called m^*-measurable if

$$(1) \qquad m^*(T) = m^*(T \cap E) + m^*(T - E)$$

for every subset T of X. In (1) we refer to T as the "test set". If there is no possibility of ambiguity, we shall say that E is <u>measurable</u> only.

The utility of the above definition was discovered by C. Caratheodory (1873-1950). He noted that a theory of measure in which $m^*(E)$ is to be the measure of E (when E is restricted to a suitable ring) demands that (1) must necessarily hold whenever T is in this ring: $T \cap E$ and $T - E$ are disjoint, and $(T \cap E) \cup (T - E) = T$ implies by property 4 of m^* that

$$m^*(T) \leq m^*(T \cap E) + m^*(T - E) .$$

Hence, E is measurable if and only if the following inequality

$$(4.3) \qquad m^*(T) \geq m^*(T \cap E) + m^*(T - E)$$

is satisfied for every subset T of X.

In the following theorems, we will show that the collection of all measurable subsets of X is a σ-ring and that m^* turns out to be a measure on $\mathcal{M}$.

4.4. THEOREM. <u>A set E is measurable if and only if its complement -E is measurable.</u>

PROOF. For any subset T of X, $T \cap E = T - (-E)$. Thus, the equation defining the measurability of E is the same as that defining the measurability of -E.

4.5. THEOREM. <u>The empty set $\emptyset$ and the whole space X are both measurable sets.</u>

PROOF. The result follows immediately from the fact that $m^*(\emptyset) = 0$.

4.6. THEOREM. <u>If E_1 and E_2 are measurable, then so are $E_1 \cup E_2$, $E_1 \cap E_2$ and $E_1 - E_2$.</u>

PROOF. Let T be any subset of X. Since E_1 is measurable by hypothesis, we have

$$(2) \qquad m^*(T) = m^*(T \cap E_1) + m^*(T - E_1) \quad .$$

Next, using $T - E_1$ as the test set for the measurability we can write

$$(3) \qquad \begin{aligned} m^*(T - E_1) &= m^*((T - E_1) \cap E_2) + m^*((T - E_1) - E_2) \\ &= m^*((T - E_1) \cap E_2) + m^*(T - (E_1 \cup E_2)) \end{aligned}$$

because E_2 is measurable. In the first term on the right in (2) we can write $T \cap E_1 = T \cap (E_1 \cup E_2) \cap E_1$. In the first

term on the right in (3) we can write
$(T - E_1) \cap E_2 = T \cap (E_1 \cup E_2) - E_1$. Substituting (3) into (2) and using property 4 of an outer measure (4.1) we have:

$$m^*(T) = m^*(T \cap (E_1 \cup E_2) \cap E_1) + m^*(T \cap (E_1 \cup E_2) - E_1) + m^*(T - (E_1 \cup E_2))$$

$$= m^*(T \cap (E_1 \cup E_2)) + m^*(T - (E_1 \cup E_2)) .$$

Therefore $E_1 \cup E_2$ is measurable.

To prove that $E_1 \cap E_2$ is measurable, we observe that $E_1 \cap E_2 = -((-E_1) \cup (-E_2))$, then by theorem 4.4, we achieve as claimed.

Finally, $E_1 - E_2$ is also measurable, because $E_1 - E_2 = E_1 \cap (-E_2)$.

We proceed next to the examination of countable unions of measurable sets.

4.7. THEOREM. <u>If</u> $\{E_k\}$ <u>is a disjoint countable family of measurable sets, and if</u> $S = \bigcup_{i=1}^{\infty} E_i$, <u>then</u> S <u>is a measurable set</u>. <u>Moreover, if</u> T <u>is any subset of</u> X, <u>then</u>

$$m^*(T \cap S) = \sum_{i=1}^{\infty} m^*(T \cap E_i)$$

PROOF. With $T \cap (E_1 \cup E_2)$ as the test set, and E_2 is measurable, we can write:

$$m^*(T \cap (E_1 \cup E_2) = m^*(T \cap (E_1 \cup E_2) \cap E_2) + m^*(T \cap (E_1 \cup E_2) - E_2)$$

$$= m^*(T \cap E_2) + m^*(T \cap E_1) \ .$$

By this method of argument and mathematical induction, we can prove that $S_n = \bigcup_{i=1}^{n} E_i$. Then

$$m^*(T \cap S_n) = \sum_{i=1}^{n} m^*(T \cap E_i)$$

Therefore, we have proved that the result holds when n is finite.

Next, for each n, $T \cap S$ contains $T \cap S_n$. Thus, for every n,

$$m^*(T \cap S) \geq m^*(T \cap S_n) = \sum_{i=1}^{n} m^*(T \cap E_i)$$

Letting n tend to infinity, we have

$$m^*(T \cap S) \geq \sum_{i=1}^{\infty} m^*(T \cap E_i) \ .$$

The inverse inequality follows from property 4 of the definition of an outer measure (4.1).

Finally, we prove that S is a measurable set. By theorem (4.6) and induction, it is clear that $S_n = \bigcup_{i=1}^{n} E_i$ is measurable. Note that $T - S_n$ contains $T - s$, for each n.

$$m^*(T) = m^*(T \cap S_n) + m^*(T - S_n)$$

$$= \sum_{i=1}^{n} m^*(T \cap E_i) + m^*(T - S_n)$$

$$\geq \sum_{i=1}^{n} m^*(T \cap E_i) + m^*(T - S)$$

Letting $n \to \infty$ and using property 4 of an outer measure, since $\bigcup_i (T \cap E_i) = T \cap S$, we obtain

$$m^*(T) \geq \sum_{i=1}^{\infty} m^*(T \cap E_i) + m^*(T - S)$$

$$\geq m^*(T \cap S) + m^*(T - S) .$$

By (4.3), S is measurable.

4.8. THEOREM. <u>The collection</u> $\mathcal{M}$ <u>of all measurable subsets of</u> X <u>is a</u> σ-<u>algebra and</u> m* <u>is a measure on</u> $\mathcal{M}$.

PROOF. We know that $\mathcal{M}$ is a ring by theorem 4.6. To prove that it is a σ-ring, suppose the sequence $\{E_i\}$ is in $\mathcal{M}$. Let $F_1 = E_1$ and let $F_n = E_n - (E_1 \cup \ldots \cup E_{n-1})$ if $n \geq 2$. Then $\{F_n\}$ is a disjoint family of measurable sets satisfying

$\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n$. Theorem 4.7 we have that $\mathcal{M}$ is a σ-ring because any countable union of subsets in $\mathcal{M}$ is again in $\mathcal{M}$. Next, we prove that m* is countably additive on $\mathcal{M}$. By taking T = S in theorem 4.7, we have

$$m^*(S) = \sum_{i=1}^{\infty} m^*(S \cap F_i) = \sum_{i=1}^{\infty} m^*(F_i) .$$

Hence m* is a measure on $\mathcal{M}$.

When we say m* is a measure on $\mathcal{M}$, what we mean is that the restriction of m* to $\mathcal{M}$ is a measure on $\mathcal{M}$. This measure, whose definition is $m(E) = m^*(E)$, if E is in $\mathcal{M}$, is called <u>the measure induced by</u> m* .

The following is an important theorem.

4.9. THEOREM. <u>Every set</u> E <u>for which</u> $m^*(E) = 0$ <u>is measurable</u>. <u>As a sequence, if</u> F <u>is a subset of</u> E <u>and</u> $m^*(E) = 0$, <u>then</u> F <u>is measurable</u>.

PROOF. For any set T, $T \cap E \subset E$ and $T - E \subset T$. Then by property 3 of an outer measure, $m^*(T \cap E) \le m^*(E)$ and $m^*(T - E) \le m^*(T)$. If $m^*(E) = 0$, then $m^*(T \cap E) = 0$, and also

$$m^*(T \cap E) + m^*(T - E) \le m^*(T) .$$

As we remarked earlier, (4.3) shows that E is measurable.

If $F \subset E$, then $m^*(F) \le m^*(E) = 0$, by the first part of this theorem, F is measurable because $m^*(F) = 0$.

From the concept of a measure on the σ-ring $\mathcal{M}$ of measurable subsets of X, and the additional condition of a continuous function, we will present a new function called measurable function in the next section.

5. MEASURABLE FUNCTIONS

Let X be a nonempty set, and let $\mathcal{M}$ be a nonempty σ-ring of subsets of X. It is assumed that we have a measure m defined on $\mathcal{M}$. The elements of $\mathcal{M}$ will be referred to as <u>measurable set</u> (or as m-measurable sets). Now we call that X is said to be a <u>measure space</u> we are thus provided with a σ-ring $\mathcal{M}$ of subsets of X and a measure m on $\mathcal{M}$. Strictly speaking it is not X alone, but the triple $(X, \mathcal{M}, m)$ which is the measure space.

If X is a metric space, then we recall that a continuous function f is characterized by the condition that for every open set G in the range space, $f^{-1}(G)$ is open in the domain. By the ways of similar characterizations of inverse images, we define two other important classes of functions.

5.1. DEFINITION. Let X be a metric space, and let f be a finite, real-valued function whose domain is the class of Borel sets in X. We say that f is a <u>Baire Function</u> if for every open set G in the real number system, $f^{-1}(G)$ is a Borel Set.

5.2. DEFINITION. Let X be any space in which an outer measure is defined, and let f be a finite, real-valued function whose domain is a measurable set in X. We say that f is a <u>measurable function</u> if for every open set G in the real number system, $f^{-1}(G)$ is a measurable set.

In order to extend these last two concepts to the case of a function which assumes infinite values we add the requirement that $f^{-1}(+\infty)$ and $f^{-1}(-\infty)$ (i.e., we mean $f^{-1}(\{\pm\infty\})$) be Borel Sets or measurable sets.

Note that the notions of a continuous function and a Baire function depend on topological properties of inverse images, while a measurable function depends only on a measure theoretic property of inverse images.

5.3. THEOREM. <u>Let</u> X <u>be</u> <u>a</u> <u>metric</u> <u>space</u>, <u>let</u> $f:X \to R$ <u>be</u> <u>a</u> <u>continuous</u> <u>function</u>, <u>and</u> <u>let</u> m* <u>be</u> <u>an</u> <u>outer</u> <u>measure</u> <u>on</u> σ-<u>ring</u> $\mathcal{M}$. <u>Then</u> <u>the</u> <u>following</u> <u>statements</u> <u>are</u> <u>true</u>:

(1) <u>The</u> <u>function</u> f <u>is</u> <u>a</u> <u>Baire</u> <u>function</u>

(2) <u>If</u> m* <u>is</u> <u>a</u> <u>metric</u> <u>outer</u> <u>measure</u>, <u>then</u> <u>every</u> <u>Baire</u> <u>function</u> <u>is</u> <u>measurable</u>

(3) <u>If</u> <u>every</u> <u>continuous</u> f <u>on</u> X <u>is</u> <u>measurable</u>, <u>then</u> m* <u>is</u> <u>a</u> <u>metric</u> <u>outer</u> <u>measure</u>.

PROOF. (1) Let $\mathcal{G}$, $\mathcal{B}$ and $\mathcal{M}$ be, respectively, the classes of open, Borel, and measurable sets in X. We alwyas have that $\mathcal{G} \subset \mathcal{B}$. So if f is continuous, and G is an open set in the real line, then $f^{-1}(G) \in \mathcal{G}$; thus f is a Baire function.

(2) Next, if m^* is a metric outer measure, then $\mathcal{B}$ is a subset of $\mathcal{M}$; and a similar argument shows that every Baire function is measurable.

(3) Finally, suppose every continuous fucntion is measurable. Let G be an arbitrary open set in X, and define F as follows:

$$f(x) = d(x;-G) \quad \text{where } -G \text{ is the complement of } G.$$

Thus, f is continuous because for every $\varepsilon > 0$, there exists $\delta = \varepsilon > 0$ such that $d(x,y) < \varepsilon$ implies

$$d[f(x),f(y)] = |f(x)-f(y)| = |d(x,-G) - d(y,-G)|$$
$$\leq |d(x,y) + d(y,-G) - d(y,-G)| < \varepsilon.$$

Hence f is measurable, and since $(0,\infty)$ is an open set in the real line, then $f^{-1}((0,\infty)) = G$ must be a measurable set. That is, every open set in X is measurable.

Now, if $d(A,B) > 0$, there is an open set G such that G contains A and -G contains B. Since G is a m^*-measurable set, by definition 5.2, we have

$$m^*(A \cup B) = m^*((A \cup B) \cap G) + m^*((A \cup B) - G)$$
$$= m^*(A) + m^*(B)$$

This shows that m^* is a metric outer measure.

In general, there may be no particular relationship between the class of measurable functions and that of Baire functions or that of Continuous functions. However, from the above result, we can think of these concepts in an ordered list: Continuous functions, Baire functions, and measurable functions.

A more important role played by the Baire functions in the theory of measurable functions is indicated by the next two results.

5.4. THEOREM. <u>If</u> f <u>is a measurable function on</u> $\mathcal{M}$ <u>and if</u> B <u>is a Borel set in the real number system, then</u> $f^{-1}(B)$ <u>is a measurable set in</u> X.

PROOF. Let $\mathcal{E}$ be the class of all sets in the real number system whose inverse images under f are measurable.

If A is in $\mathcal{E}$, then

$$f^{-1}(-A) = -f^{-1}(A)$$

is measurable by Theorem 4.4. Thus -A is in $\mathcal{E}$.

If $E = \bigcup_{i=1}^{\infty} E_i$ then by the properties of inverse images we have

$$f^{-1}\left(\bigcup_{i=1}^{\infty} E_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(E_i) .$$

By theorem 4.8, since $\mathcal{M}$ is a σ-ring then $\bigcup_{i=1}^{\infty} f^{-1}(E_i)$ is in $\mathcal{E}$. By Definition 5.2, we know that $\mathcal{E}$ contains all of the open sets, therefore it contains all Borel sets since Borel sets is the smallest σ-algebra which contains all of the open sets.

∎

5.5. COROLLARY. *If f is a measurable function on any space X, and if g is a Baire function on the real number system, then the composite function* $g \circ f$ *is a measurable function on X.*

PROOF. Let G be any open set in the real line; since

$$(g \circ f)^{-1}(G) = f^{-1}(g^{-1}(G))$$

and since $g^{-1}(G)$ is a Borel set; by the Theorem, $f^{-1}(g^{-1}(G))$ is a measurable set in X; that is, $g \circ f$ is a measurable function on X.

∎

5.6. THOEREM. *Let f be an extended real-valued function whose domain is measurable. Then the following statements are equivalent:*

1) *For each real number a, the set* $\{x : f(x) > a\}$ *is measurable*
2) *For each real number a, the set* $\{x : f(x) \geq a\}$ *is measurable*
3) *For each real number a, the set* $\{x : f(x) < a\}$ *is measurable*
4) *For each real number a, the set* $\{x : f(x) \leq a\}$ *is measurable.*

These statements imply

(5) *For each extended real number* a, *the set* $\{x|f(x)=a\}$ *is measurable.*

PROOF. Let D be the domain of f. We have (1) implies (4): since $\{x:f(x) \leq a\} = D - \{x:f(x) > a\}$, and the difference of two measurable sets is a measurable set. Similarly, (4) implies (1) and (2) is equivalent to (3);

(1) implies (2): Since $\{x:f(x) \geq a\} = \bigcap_{n=1}^{\infty} \{x:f(x) > a - \frac{1}{n}\}$

and the intersection of any sequence of measurable sets is measurable (by the result of Theorem 4.4 and Theorem 4.8), we obtain (2) from (1).

Similarly, (2) implies (1) since

$$\{x:f(x) > a\} = \bigcup_{n=1}^{\infty} \{x:f(x) \geq a + \frac{1}{n}\}.$$

Thus the first four statements are equivalent. Next we show that each implies (5):

If a is any real number, then

$$\{x:f(x) = a\} = \{x:f(x) \geq a\} \cap \{x:f(x) \leq a\}$$

and so (2) and (4) imply (5). Moreover, since $\{x:f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x:f(x) \geq n\}$, (2) gives (5) for a positive infinite. Similarly, (4) implies (5) for $a = -\infty$.

It should be noted that a continuous function (with a measurable domain) is measurable. The notion of a measurable function is quite analogous to that of a continuous function. However, the following characterization of a measurable function is very useful and is frequently given as the definition.

5.7. THEOREM. _In order that an extended real-valued function f defined on a measurable set D be a measurable function, it is necessary and sufficient that for every finite real number a, $f^{-1}\{[-\infty,a]\}$ be a measurable set._

PROOF. To prove the necessity, we note that $f^{-1}\{[-\infty,a]\} = D - (f^{-1}\{(a,\infty)\} \cup f^{-1}(\infty))$. If f is measurable, then each set on the right hand side is measurable because (a,∞) is an open set and $f^{-1}(\infty)$ is a measurable set. Thus $f^{-1}\{[-\infty,a]\}$ is measurable.

Conversely, if $f^{-1}\{[-\infty,a]\}$ is a measurable set for each a, then each of the sets

$$f^{-1}(-\infty) = \bigcap_{n=1}^{\infty} f^{-1}\{[-\infty,-n]\} \quad ; \quad f^{-1}(+\infty) = D - \bigcup_{n=1}^{\infty} f^{-1}\{[-\infty,n]\}$$

is measurable.

For any half-open interval $(a,b] = [-\infty,b] - [-\infty,a]$

$$f^{-1}\{(a,b]\} = f^{-1}\{[-b,b]\} - f^{-1}\{[-\infty,a]\}$$

is measurable. Next, if G is any open set in the real line

then we claim that G can be expressed as a countable union of disjoint half-open intervals $(a_i, b_i]$, $i = 1,2,\ldots$

If this is true, it follows immediately that

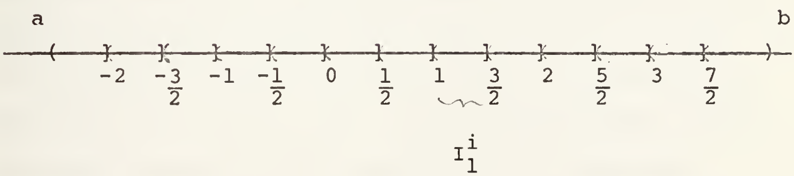
$$f^{-1}(G) = \bigcup_{i=1}^{\infty} f^{-1}\{(a_i, b_i]\}$$

is a measurable set.

To prove our claim, let G be any open set in the real line, say $G = (a,b)$. For each integer k, the points x's such that

$$(1) \qquad x = \frac{m}{2^k} \quad (m = \ldots,-2,-1,0,1,2,\ldots)$$

partition R into a countable class of disjoint half-intervals. For example, when $k = 1$, let I_1^1 , I_1^2 , I_1^3 ,... be the class of intervals generated by (1) which are contained in G.



For each $k > 1$, let I_k^1 , I_k^2 , I_k^3 ,... be that class of intervals generated by (1) which are contained in G but not contained in any interval I_q^p with $q < k$, or we can say a partition $P_k = \{I_k^i : i=1,2,3,\ldots\}$ is a refinement of the partition P_q if $q < k$. If x in G, then x is an interior point of G; so there is a partition of R given by (1) such that

the interval containing x is contained in G. Thus

$$\bigcup_{k=1}^{\infty} \bigcup_{j} I_k^j \supset G \quad .$$

Since $I_k^j \subset G$ for each k,j ; we have

$$\bigcup_{k=1}^{\infty} \bigcup_{j} I_k^j \subset G \quad .$$

This is clearly a countable class of half-open intervals, and we have constructed them so that they are disjoint.

6. OPERATIONS ON MEASURABLE FUNCTIONS

Let f be an extended real-valued function on X, and let a be any real number. Each of these functions: af, a + f, $|f|^a$, f^+ , and f^- is defined everywhere f is, with the exceptions that if $a = 0$, af and $|f|^a$ are not defined at points x for which $|f(x)| = 0$; and if $a < 0$, $|f|^a$ is not defined at points x for which $f(x) = 0$ because such operations as $0 \cdot (+\infty)$, ∞^0 , $\frac{1}{0}$ are not sensible. These exceptions are not serious because if f is measurable, $\{x: f(x) < \infty\}$ and $\{x: f(x) \neq 0\}$ are measurable; so af and $|f|^a$ always have measurable domains.

6.1. THEOREM. If f is a measurable function and a is any real number, then each of the functions af , a + f , $|f|^a$, f^+ , and f^- is measurable.

PROOF. Each of the functions $g:y \mapsto ay$, $a+y$, $|y|^a$, $\max\{y,0\}$, $\min\{y,0\}$ is a continuous function of y. By result 1) from Theorem 5.3 we know that a continuous function is a Baire function. The function $x \mapsto f(x) \mapsto af(x)$, $a+f(x)$, $|f(x)|^a$, $f^+(x)$, $f^-(x)$ are the composite of function g's and measurable f, then by corollary 5.5, af, $a+f$, $|f|^a$, f^+ and f^- are measurable functions as claimed.

The function $f+g$ is undefined on $\{x:f(x) = -g(x) = \pm\infty\}$ and fg is undefined on

$$\{x:f(x) = 0;g(x) = \pm\infty\} \quad \{x:f(x) = \pm\infty;g(x) = 0\} .$$

However, for f and g measurable, each of these is a measurable set, so $f+g$ and fg have measurable domains.

6.2. LEMMA. <u>If f and g are measurable functions then $\{x:f(x) \leq g(x)\}$ is a measurable set.</u>

PROOF. For a given x, $f(x) > g(x)$ if and only if there is a rational number r such that $g(x) < r$ and $r < f(x)$. Hence if $\{r_n\}$ is an enumeration of the rationals, we see that

$$\{x:f(x) > g(x)\} = \bigcup_{n=1}^{\infty} [\{x:f(x) > r_n\} \cap \{x:g(x) < r_n\}] .$$

Since f and g are measurable functions then $\{x:f(x) > g(x)\}$ is a measurable set by Theorems 5.7 and 4.8.

6.3. THEOREM. *If f and g are measurable functions then $f+g$ is a measurable function.*

PROOF. The proof of Theorem 6.3 is as follows:

$$\{x : f(x)+g(x) \leq a\} = \{x : f(x) \leq a - g(x)\} .$$

By the result of 6.1, $a-g$ is a measurable function. Thus, by Lemma 6.2, $\{x : f(x)+g(x) \leq a\}$ is a measurable set, and by 5.7, $f+g$ is a measurable function.

By the previous results, we see that the set of measurable functions on X is a real vector space.

6.4. COROLLARY. *In order that f be measurable, it is necessary and sufficient that both f^+ and f^- be measurable.*

PROOF. Necessity of this condition is included in 6.1. Sufficiency comes from 6.3 and the fact that $f = f^+ - f^-$.

6.5. COROLLARY. *If f and g are measurable, then fg is measurable.*

PROOF. The function f^2 is measurable, since

$$\{x : f^2(x) > a\} = \{x : f(x) > \sqrt{a}\} \cup \{x : f(x) < -\sqrt{a}\}$$

for $a > 0$, and for $a < 0$ we have $\{x : f^2(x) > a\} = D$ where D is the domain of f.

Thus, by the fact that $fg = \frac{1}{4}((f+g)^2 - f^2 - g^2)$ it follows that fg is measurable.

Let f and g be measurable functions. From the equations $f \vee g = \frac{1}{2}(f + g + |f-g|)$ and $f \wedge g = \frac{1}{2}(f + g - |f-g|)$ it follows that $f \vee g$ and $f \wedge g$ are measurable functions. Thus, the set of measurable functions on X is a vector lattice, and also an algebra.

7. SEQUENCES OF MEASURABLE FUNCTIONS

Next, we are going to consider some properties of sequences of measurable functions. We will see that the limit function of an a.e. convergent sequence of measurable functions is again measurable.

7.1. THEOREM. <u>If $\{f_n\}$ is a sequence of measurable functions on some common domain, then</u> $\sup_n f_n$, <u>and</u> $\inf_n f_n$, $\overline{\lim_n} f_n$ <u>and</u> $\underline{\lim}_n f_n$ <u>are measurable functions.</u>

PROOF. From the fact that

$$\{x : \sup_n f_n(x) \leq a\} = \bigcap_{n=1}^{\infty} \{x : f_n(x) \leq a\} .$$

By Theorem 5.6, then this gives the measurability of $\sup_n f_n$. The result for $\inf_n f_n$, follows from the fact that $\inf_n f_n = -\sup_n (-f_n)$ and Theorem 6.1.

According to the definitions:

$$\overline{\lim_n} f_n = \inf_k \sup_{n \geq k} f_n \quad \text{and} \quad \underline{\lim}_n f_n = \sup_k \inf_{n \geq k} f_n ,$$

by the first part of this Theorem, it follows that $\limsup_n f_n$ and $\liminf_n f_n$ are measurable functions completing the proof.

∎

7.2. THEOREM. If f is a measurable function and $f = g$ a.e., then g is measurable.

PROOF. Let $E = \{x : f(x) \neq g(x)\}$. By hypothesis, $m(E) = 0$. Now,

$$\{x:g(x) > a\} = (\{x:f(x) > a\} \cup \{x\varepsilon E:g(x) > a\}) - \{x\varepsilon E:g(x) \leq a\}$$

Since f is measurable, then the first set on the right is measurable. The last two sets form a partition of E and $m(E) = 0$; thus they are measurable. Therefore, $\{x:g(x) > a\}$ is measurable for each a, so g is a measurable function.

∎

7.3. COROLLARY. If $\{f_n\}$ is a sequence of measurable functions on some common domain D and if $\lim f_n = f$ a.e., then f is a measurable function.

PROOF. Let $E = \{x:\lim f_n(x) \neq f(x)\}$, then $m(E) = 0$. The values of f for x in E are insignificant, since every subset of a null set is measurable. For x in $D - E$, since $\{f_n\}$ converges, then $f = \overline{\lim_n} f_n$, the result follows from 7.1.

∎

The corollary 7.3 tells us that the class of measurable functions on X is closed under an even more general operation: that of taking a.e. limits. We know that the class of

continuous functions on X is closed under the operation of taking uniform limits, but not under that of taking pointwise limits.

In the following section, we will construct the theory of a class of integrable functions on a measurable set via the theory of measurable functions and measurable sets.

8. INTEGRATION

One of the simplest interpretations of a definite integral is that of the area under a curve. More generally, in this chapter, integral generated by a measure function, that is a generalization of the notion of area. We define the integral of a simple function as the sum of the "areas" under its constant sections and then, we extend by taking limits to get the integral of a more general function.

First of all, we will show that any nonnegative, measurable function can be expressed as a limit function of a nondecreasing sequence of nonnegative simple functions.

Recall that the characteristic function χ_E of a set E is defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if} \quad x \varepsilon E \\ 0 & \text{elsewhere} . \end{cases}$$

Thus, the function χ_E is measurable if and only if E is a measurable set.

8.1. DEFINITION. Let $E_1, E_2, \ldots, E_n$ be any finite class of disjoint, measurable sets, and let $a_1, a_2, \ldots, a_n$ be any corresponding set of real numbers. A function $f: X \to R$ defined by

$$f(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$$

is called a <u>simple function</u>.

Informally, a simple function is one which assumes a finite number of values and assumes each of these values on a measurable set. Clearly, every simple function is bounded and measurable.

Throughout this section the set E will always refer to a measurable set.

8.2. THEOREM. <u>If f is a nonnegative, measurable function on a set</u> E, <u>then there exists a nondecreasing sequence</u> $\{f_n\}$ <u>of nonnegative, simple functions such that</u> $\lim f_n(x) = f(x)$ <u>for every</u> $x \in E$.

PROOF. For each integer n, define the function given by:

$$f_n(x) = \begin{cases} \dfrac{i-1}{2^n} & \text{for} \quad \dfrac{i-1}{2^n} \leq f(x) < \dfrac{i}{2^n} \ ; \quad i=1,2,\ldots,n2^n \ . \\ \\ n & \text{for} \quad f(x) \geq n \end{cases}$$

for each $x \in E$.

For example, let us look at f_1 and f_2 of this sequence. For each $x \varepsilon E$, we have f_1:

For $i = 1$;

$$f_1(x) = \begin{cases} 0 & \text{if} \quad 0 \leq f(x) < \frac{1}{2} \\ 1 & \text{if} \quad f(x) \geq 1 \end{cases}$$

For $i = 2$;

$$f_1(x) = \begin{cases} \frac{1}{2} & \text{if} \quad \frac{1}{2} \leq f(x) < 1 \\ 1 & \text{if} \quad f(x) \geq 1 \end{cases}$$

For $i \geq 3$;

$$f_1(x) = 1$$

and f_2; for each $x \varepsilon E$:

$i = 1$;

$$f_2(x) = \begin{cases} 0 & \text{if} \quad 0 \leq f(x) < \frac{1}{4} \\ 2 & \text{if} \quad f(x) \geq 2 \end{cases}$$

$i = 2$;

$$f_2(x) = \begin{cases} \frac{1}{4} & \text{if} \quad \frac{1}{4} \leq f(x) < \frac{1}{2} \\ 2 & \text{if} \quad f(x) \geq 2 \end{cases}$$

$i = 3$;

$$f_2(x) = \begin{cases} \frac{1}{2} & \text{if} \quad \frac{1}{2} \leq f(x) < \frac{3}{4} \\ 2 & \text{if} \quad f(x) \geq 2 \end{cases}$$

$i = 4$;

$$f_2(x) = \begin{cases} \frac{3}{4} & \text{if} \quad \frac{3}{4} \leq f(x) < 1 \\ 2 & \text{if} \quad f(x) \geq 2 \end{cases}$$

$i = 5;$

$$f_2(x) = \begin{cases} 1 & \text{if} \quad 1 \leq f(x) < \frac{5}{4} \\ 2 & \text{if} \quad f(x) \geq 2 \end{cases}$$

$i = 6;$

$$f_2(x) = \begin{cases} \frac{5}{4} & \text{if} \quad \frac{5}{4} \leq f(x) < \frac{6}{4} \\ 2 & \text{if} \quad f(x) \geq 2 \end{cases}$$

$i = 7;$

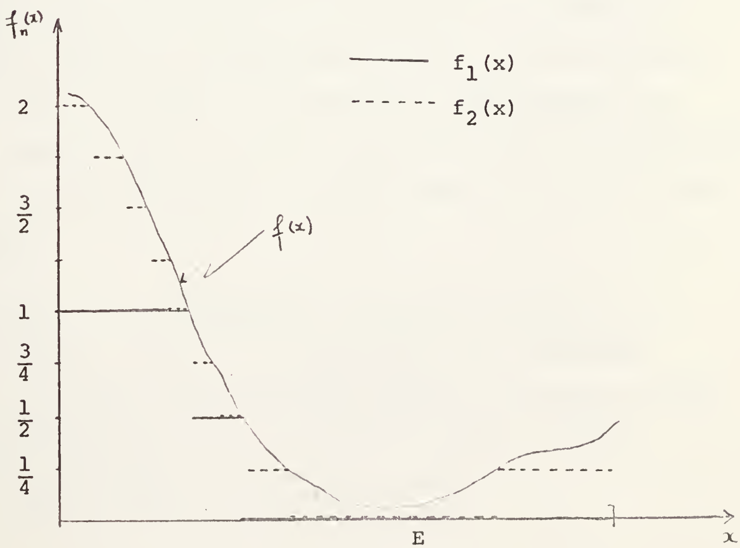
$$f_2(x) = \begin{cases} \frac{6}{4} & \text{if} \quad \frac{6}{4} \leq f(x) < \frac{7}{4} \\ 2 & \text{if} \quad f(x) \geq 2 \end{cases}$$

$i = 8;$

$$f_2(x) = \begin{cases} \frac{7}{4} & \text{if} \quad \frac{7}{4} < f(x) \leq 2 \\ 2 & \text{if} \quad f(x) > 2 \end{cases}$$

$i \geq 9$

$$f_2(x) = 2$$



It is clear that each function f_n is simple and nonnegative. Look at f_1 and f_2 above; we always have $f_2(x) \geq f_1(x)$ for all $x \in E$. By the same argument and by induction, we have that the sequence $\{f_n\}$ is nondecreasing.

If $f(x) < \infty$, then for $n > f(x)$, we have

$$0 \leq f(x) - f_n(x) < \frac{1}{2^n} .$$

For $f(x) = \infty$, then for each n, $f_n(x) = n$. So, in either case, we have

$$\lim f_n(x) = f(x) \qquad \text{for every } x \text{ in } E.$$

∎

In the next several sections, our development of the integral is based on monotone sequences of simple functions, and in this connection Theorem 8.2 is vital. It tells us that every nonnegative, measurable function is the pointwise limit of such a sequence. The following embellishments to 8.2 will also be used to develop the definition of the integral.

The following theorem is a useful result in order to prove the others.

8.3 THEOREM. Let f be a nonnegative, measurable function on a set E. The set $\{x:f(x) \neq 0\}$ is the union of a countable class of measurable sets of finite measure if and only if for each simple function f_n, $\{x:f_n(x) \neq 0\}$ has a finite measure, and for each x, $\lim f_n(x) = f(x)$.

PROOF. In order to prove the first statement, let

$$\{x:f(x) \neq 0\} = \bigcup_{i=1}^{\infty} E_i ,$$

and let

$$A_n = \bigcup_{i=1}^{n} E_i \quad \text{for each } n.$$

Then $\{A_n\}$ is a nondecreasing sequence of measurable sets, so the characteristic functions χ_{A_n} form a nondecreasing sequence of functions. If $\{f_n\}$ is the sequence constructed as the proof of 8.2, then the sequence $\{g_n\}$ defined by

$$g_n = f_n \chi_{A_n}$$

is a nondecreasing sequence of simple and nonnegative functions the same as $\{f_n\}$ and $\{\chi_{A_n}\}$ and for each n,

$$\{x:g_n(x) \neq 0\} = \{x:f_n(x) \neq 0\} \cap \{x:\chi_{A_n}(x) \neq 0\}$$

is contained in A_n which has finite measure.

To prove the converse, we know that, if $f_n(x) = 0$ for every n, then $\lim f_n(x) = f(x) = 0$. So we have the contrapositive inclusion:

$$\{x:f(x) \neq 0\} \subset \bigcup_{n=1}^{\infty} \{x:f_n(x) \neq 0\} .$$

Setting $E_n = \{x:f(x) \neq 0\} \cap \{x:f_n(x) \neq 0\}$. Thus

$$\{x:f(x) \neq 0\} \cap \left[\bigcup_{n=1}^{\infty} \{x:f_n(x) \neq 0\}\right] = \bigcup_{n=1}^{\infty} E_n ,$$

so we have $$\{x:f(x) \neq 0\} = \bigcup_{n=1}^{\infty} E_n .$$

The sets E_n are measurable and each has finite measure. ∎

Now we have the concept of the integral of a simple function.

Let $f:X \to R$ be a simple function with distinct values $a_1,\ldots,a_n$. Let $E_i = \{x:f(x) = a_i\}$. Then the E_i's are disjoint, measurable sets, and their union is X. Moreover, by the definition of a simple fucntion we have

$$f = \sum_{i=1}^{n} a_i \chi_{E_i} .$$

Let f be a simple function, and let E be a measurable set. We say that f is <u>integrable</u> on E if

$$m(E \cap \{x:f(x) \neq 0\}) < \infty .$$

Clearly, the definition is equivalent to the condition that if

$$(8.4) \qquad f = \sum_{i=1}^{n} a_i \chi_{E_i}$$

where $a_i \neq 0$, $i=1,2,\ldots,n$ then $m\left(\bigcup_{i=1}^{n} (E \cap E_i)\right) < \infty$.

If a simple function is integrable on X, we say merely that f is <u>integrable</u>. Obviously, if a simple function is integrable, then it is integrable on every measurable set $E \subset X$. If the simple function f described by (8.4) is integrable on a measurable set E, we define

$$(8.5) \qquad \int_E f = \sum_{i=1}^{n} a_i \, m(E \cap E_i) \ .$$

It follows easily from the additivity of m that if f is also equal to $\sum_{i=1}^{m} b_i \chi_{F_i}$, then $\int_E f = \sum_{i=1}^{m} b_i \, m(E \cap F_i)$, i.e. that the value of the integral is independent of the presentation of f, and therefore unambiguously defined.

From the finite additive property of a measure m, we will have a useful property of integral of a simple function over the disjoint measurable sets as follows: With fixed f, we can consider $\int_E f$ as the value of a set function defined on the σ-ring $\mathcal{M}$, for all $E \in \mathcal{M}$.

8.6. THEOREM. <u>Let</u> $E_1, E_2, \ldots, E_n$ <u>be disjoint measurable sets with</u>

$$E = \bigcup_{i=1}^{n} E_i$$

<u>and let</u> f <u>be a simple function which is integrable on each set</u> E_i, <u>then</u> f <u>is integrable on</u> E, <u>and</u>

$$\int_E f = \sum_{i=1}^{n} \int_{E_i} f$$

PROOF. Let $A = \{x : f(x) \neq 0\}$.

Since f is integrable on E_i , $i=1,2,\ldots,n$, then

$$m(E \cap A) = \sum_{k=1}^{n} m(E_k \cap A) < \infty$$

So f is integrable on E by definition.

Let

$$f = \sum_{j=1}^{m} a_j \chi_{A_j}$$

By (8.5) we can write

$$\sum_{i=1}^{n} \int_{E_i} f = \sum_{i=1}^{n} \sum_{j=1}^{m} a_j\, m(E_i \cap A_j) = \sum_{j=1}^{m} a_j \sum_{i=1}^{m} m(E_i \cap A_j)$$

$$= \sum_{j=1}^{m} a_j\, m(E \cap A_j) = \int_E f \quad .$$

∎

We turn now to a consideration of the basic properties of the set of simple functions.

8.7. THOEREM. The set of simple functions that are integrable on E forms a real linear space, and the integral on E is a linear functional on this set.

PROOF. Let g and f be simple functions that are integrable on E, and let a and b be arbitrary real numbers.

First observe that

$$\{x : af(x) + bg(x) \neq 0\} \subset \{x : f(x) \neq 0\} \cup \{x : g(x) \neq 0\}$$

Since $m(E \cap \{x : f(x) \neq 0\}) + m(E \cap \{x : g(x) \neq 0\}) < \infty$, it follows from the above observation that $af + bg$ is integrable on E.

Next let

$$f = \sum_{i=1}^{n} a_i \chi_{A_i} \quad \text{and} \quad g = \sum_{j=1}^{m} b_j \chi_{B_j}$$

Then for each i and j, it is the case that f and g are both constant on $A_i \cap B_j$; $i=1,2,\ldots,n$ and $j=1,2,\ldots,m$. If $x \in A_i \cap B_j$, then

$$f(x) = a_i \quad \text{and} \quad g(x) = b_j$$

Thus,

$$(af+bg)(x) = \begin{cases} aa_i + bb_j & \text{for } x \in A_i \cap B_j \cap E \\ 0 & \text{for } x \text{ not in } E. \end{cases}$$

Therefore,

$$\int_{E \cap A_i \cap B_j} (af + bg) = (aa_i + bb_j)\, m(E \cap A_i \cap B_j)$$

$$= aa_i\, m(E \cap A_i \cap B_j) + bb_j\, m(E \cap A_i \cap B_j)$$

$$= \int_{E \cap A_i \cap B_j} f + b \int_{E \cap A_i \cap B_j} g .$$

Since $(af+bg)(x) = 0$ if x is not in E, and

$$X = \bigcup_{i=1}^{n} \bigcup_{j=1}^{n} (A_i \cap B_j) \supset E ,$$

the result follows from Theorem 8.6. ∎

We are going to find that a simple function is integrable on E if it is bounded by an integrable simple function.

8.8. THEOREM. <u>If the simple functions</u> f <u>and</u> $g: X \to R$, <u>if</u> f <u>is integrable on</u> E, <u>and if for each</u> x <u>in</u> E, $|g(x)| \leq f(x)$, <u>then</u> g <u>is integrable on</u> E, <u>and</u>

$$\int_E g \leq \int_E f$$

PROOF. Let $A = \{x: g(x) \neq 0\}$ and $B = \{x: f(x) \neq 0\}$.

For x in A, $g(x) \neq 0$ and $|g(x)| \leq f(x)$. Thus $f(x) \neq 0$. Hence, $A \subset B$. It follows that $E \cap A \subset E \cap B$; thus $m(E \cap A) \leq m(E \cap B) < \infty$. Therefore g is integrable on E by definition.

Now let $h = f - g$, then h is nonnegative simple function. It is obvious from the definition of the integral of a simple function that $\int_E h \geq 0$.

Also by Theorem 8.7

$$\int_E f - \int_E g = \int_E (f-g) = \int_E h \geq 0 .$$

Thus $$\int_E f \geq \int_E g .$$ ∎

The following results are required for our general discussion of integration in the next section. Moreover, the next theorem shows that the set of integrable simple functions on E and the integral on that set of simple functions possess the second property required for the Daniell development of integration.

8.9 THEOREM. Let $\{f_n\}$ be a nonincreasing sequence of nonnegative simple functions that are integrable on E, and let $\lim f_n(x) = 0$ for each x in E. Then $\lim \int_E f_n = 0$.

PROOF. Let $F = \{x \varepsilon E : f_1(x) \neq 0\}$ and $M = \max\{f_1(x) : x \text{ in } E\}$. Evidently $f_n \leq M\,\chi_F$ for all n.

Given any $\varepsilon > 0$, define $E_n = \{x : f_n(x) \geq \varepsilon\}$. We assert that $\lim m(E_n) = 0$. In any case it is clear that E_n is a subset of F. We see that $\lim m(E_n) = 0$ results from the assumption that $\lim f_n(x) = 0$ for each x. Since E_n have finite measure, it follows from Theorem 3.7 that $\{E_n\}$ is a nonincreasing sequence of measurable sets, with $\lim m(E_n) = 0$.

Since $\{x : f_n(x) \neq 0\}$ is a subset of F we have

$f_n = \chi_F f_n = \chi_{F-E_n} f_n + \chi_{E_n} f_n$ where $\chi_F f_n(x) = \chi_F(x) \cdot f_n(x)$ for all x. Since $f_n(x) < \varepsilon$ on $E - E_n$ and $f_n \leq M\chi_F$,

it follows that $\chi_{F-E_n} f_n \leq \varepsilon\chi_{F-E_n} \leq \varepsilon\chi_F$ and $\chi_{E_n} f_n \leq M\chi_{E_n}$.

Hence, $f_n \leq \varepsilon\chi_F + M\chi_{E_n}$. Thus, for each n, (by 8.7 and 8.8)

$$\int_E f_n \leq \varepsilon \cdot m(F) + M \cdot m(E_n) .$$

Therefore $\lim \int_E f_n \leq \varepsilon \cdot m(F)$ since $\lim m(E_n) = 0$.

However, ε is an arbitrary number and $m(F)$ is a finite number independent of ε; thus $\lim \int_E f_n = 0$.

8.10. COROLLARY. Let $\{f_n\}$ be a nondecreasing sequence of nonnegative simple functions, that are integrable on E and let f be a simple function such that for each x in E, $\lim f_n(x) = f(x)$.

Then

$$\lim \int_E f_n = \int_E f .$$

PROOF. $\{f - f_n\}$ is a nonincreasing sequence of nonnegative simple functions that are integrable on E, and $\lim(f - f_n) = 0$. By Theorem 8.9, we have

$$\lim \int_E f_n = \int_E f .$$

This corollary still holds if the sequence $\{f_n\}$ is nonincreasing because the sequence $\{f_n - f\}$ satisfies all conditions in Theorem 8.9.

8.11. COROLLARY. *If $\{f_n\}$ and $\{g_n\}$ are two nondecreasing sequences of nonnegative, integrable, simple functions such that for each x in E,*

$$\lim f_n(x) = \lim g_n(x) \ , \quad \text{then} \quad \lim \int_E f_n = \lim \int_E g_n \ .$$

PROOF. For any fixed positive integer i, we have

$$\lim f_n(x) \geq g_i(x) \quad \text{for all } x \text{ in } E.$$

Then by Corollary 8.10, $\lim \int_E f_n \geq \int_E g_i$ for all i .

Thus,
$$\lim \int_E f_n \geq \lim \int_E g_n \ .$$

By interchanging the roles of $\{f_n\}$ and $\{g_n\}$ in this argument, we get the reverse inequality, which completes the proof.

Next, we are going to define the integral of a nonnegative measurable function. The importance of these theorems tells us that the integral of a nonnegative measurable function f is independent of the choice of the nondecreasing sequence $\{f_n\}$ of nonnegative simple functions that are integrable on E such that for each x in E, $\lim f_n(x) = f(x)$.

The integral of a measurable function will be developed from the integral of integrable simple functions. This development will be accomplished in two stages.

8.12 DEFINITION. Let f be any nonnegative measurable function, and let E be any measurable set. We say that f is <u>integrable</u> on E if there exists a nondecreasing sequence $\{f_n\}$ of nonnegative simple functions, each of which is integrable on E, such that for each x in E, $\lim f_n(x) = f(x)$, and $\lim \int_E f_n < \infty$, if this is the case, we define

$$\int_E f = \lim \int_E f_n \quad .$$

This condition is a standard one in the definition of integrability. Many authors distinguish between the statements "f is integrable on E" and "$\int_E f$ is defined," thereby allowing the possibility that $\int_E f = \infty$. We will regard these statements as equivalent and so require that $\int_E f$ always be finite.

The foregoing definition calls for several comments. First, because of Theorem 8.10 for any monotone sequence $\{f_n\}$ of integrable simple functions, $\lim \int_E f_n$ exists in the extended real number system. Next, Corollary 8.11 tells us that the definition 8.12 is independent of the choice of the sequence $\{f_n\}$ because if we have two nondecreasing sequences $\{f_n\}$ and $\{g_n\}$ of nonnegative simple functions, each of which is integrable on E such that for each x in E $\lim f_n(x) = \lim g_n(x) = f(x)$ and $\lim \int_E f_n < \infty$, then we have

$$\lim \int_E f_n = \lim \int_E g_n = \int_E f$$

Therefore the definition of $\int_E f$ is well defined.

We recall that, throughout these sections "E" will always refer to a measurable set.

We extend the domain of definition of the integral to include the functions which assume negative values.

Recall that $f^+ = f \vee 0 = \max\{f,0\}$

and $f^- = f \wedge 0 = \min\{f,0\}$

Both f^+ and f^- are nonnegative and measurable from X into R with $f = f^+ - f^-$.

8.13. DEFINITION. A measurable function $f:X \to R$ is called integrable on E if f^+ and f^- are both integrable on E.

In this case, we define

$$\int_E f = \int_E f^+ - \int_E f^- .$$

It follows definition 8.13 that for any function f is integrable on E, then so is $-f$.

8.14. THEOREM. <u>If</u> f <u>is integrable on</u> E, <u>it is integrable on every measurable subset of</u> E. <u>Furthermore, if</u> f <u>is nonnegative and integrable on</u> E, <u>then if</u> $A \subset E$, <u>we have</u>

$$\int_A f \leq \int_E f .$$

PROOF. It is integrable, then first if we take f is a nonnegative simple function, both results are obvious because if $A \subset E$, we have

$$m(A \cap \{x:f(x) \neq 0\}) \leq m(E \cap \{x:f(x) \neq 0\}) < \infty .$$

Thus, if $\{f_n\}$ is a nondecreasing sequence of nonnegative, integrable, simple functions and if for any A is a subset of E, then for every n, we have

$$\int_A f_n \leq \int_E f_n \quad .$$

Thus,

$$\lim_n \int_A f_n \leq \lim_n \int_E f_n \quad ,$$

and both results are extended to arbitrary, nonnegative integrable functions. The proof of the theorem is completed by noting that in definition 8.13 integrability for any function is defined in terms of the integrability of non-negative functions.

Because of Theorem 8.14, we frequently speak of an <u>integrable function</u>, meaning integrable on X; such a function is necessarily integrable on every measurable subset of X.

9. ELEMENTARY PROPERTIES OF THE INTEGRAL

The purpose of this section is to extend the properties of the integral of simple functions to the integral of functions in general.

9.1. THEOREM. *Let $E_1, E_2, \ldots, E_m$ be disjoint measurable sets with $E = \bigcup_{n=1}^{m} E_k$, and let f be integrable on each set E_k. Then f is integrable on E, and* $\int_E f = \sum_{k=1}^{m} \int_{E_k} f$.

PROOF. If f is a nonegative, integrable function on E_k then there exists a nondecreasing sequence $\{f_n\}$ of nonnegative simple functions, each of which is integrable on E_k, such that for each x in E_k

$$\lim f_n(x) = f(x)$$

and $$\int_{E_k} f = \lim \int_{E_k} f_n \qquad , \quad k=1,\ldots,m$$

By Theorem 8.6, the result in question holds for simple functions so we have

$$\sum_{k=1}^{m} \int_{E_k} f = \sum_{k=1}^{m} \lim \int_{E_k} f_n = \lim \Big(\sum_{k=1}^{m} \int_{E_k} f_n \Big)$$

$$= \lim \int_E f_n = \int_E f \quad .$$

Therefore, the desired result holds for nonnegative functions; the general case then follows from 8.13:

$$\int_E f = \int_E f^+ - \int_E f^- \quad ,$$

where f^+ and f^- are both nonnegative integrable functions on E.

In the next theorem we will establish that the class of integrable functions over a measurable set E has the structure of a real linear space and that the integral on that class acts as a linear functional. Thus the class of integrable functions forms an important object of study from the point of view of functional analysis.

9.2. THEOREM. <u>The set of integrable functions on</u> E <u>is a real linear space, and the integral on</u> E <u>is a linear functional on the set of integrable functions on</u> E.

PROOF. If f and g are both integrable on E, and if a and b are arbitrary real numbers, we will show that $af + bg$ is integrable on E, and that

$$\int_E (af + bg) = a \int_E f + b \int_E g \ .$$

Observe that for $a \geq 0$, $(af)^+ = af^+$, and $(af)^- = af^-$; and for $a < 0$, $(af)^+ = af^-$ and $(af)^- = af^+$. From these facts and Theorem 8.7, if $\{f_n\}$ is a nondecreasing sequence of nonnegative simple functions, that are integrable on E, such that for each x in E , $\lim f_n(x) = f(x)$ and $\lim \int_E f_n < \infty$; from Theorem 8.7, a property of the limit we have

$$\int_E af = \lim \int_E af_n = a \lim \int_E f_n = a \int_E f \ .$$

Hence, we need consider only the case $a = b = 1$.

Next, we divide X into the following sets:

$$A = \{x : f(x)g(x) \geq 0\}$$

$$B = \{x : f(x) \geq 0; g(x) < 0\}$$

$$C = \{x : f(x) < 0; g(x) \geq 0\} .$$

On thes set A, $(f+g)^+ = f^+ + g^+$ and $(f+g)^- = f^- + g^-$. Then by Theorem 8.7,

$$\int_{E\cap A} (f+g) = \int_{E\cap A} \lim(f_n + g_n) = \lim \int_{E\cap A} (f_n + g_n)$$

$$= \lim \int_{E\cap A} f_n + \lim \int_{E\cap A} g_n$$

$$= \int_{E\cap A} f + \int_{E\cap A} g .$$

We divide B into two sets

$$B_1 = B \cap \{x : f(x) + g(x) \geq 0\}$$

$$B_2 = B \cap \{x : f(x) + g(x) < 0\}$$

Each of the following relations follows from Theorem 8.7 because in B_1, $(f+g) \geq 0$ and in B_2, $-(f+g) \geq 0$, so we have

$$\int_{E\cap B_1} (f+g) = \int_{E\cap B_1} \lim(f_n + g_n) = \lim \int_{E\cap B_1} (f_n + g_n)$$

$$= \int_{E\cap B_1} f + \int_{E\cap B_1} g .$$

and

$$-\int_{E\cap B_2}(f+g) = \int_{E\cap B_2} -(f+g) = \int_{E\cap B_1} \lim(-f_n - g_n)$$

$$= \lim \int_{E\cap B_2}(-f_n) + \lim \int_{E\cap B_2}(-g_n)$$

$$= \int_{E\cap B_2}(-f) + \int_{E\cap B_2}(-g) = -(\int_{E\cap B_2} f + \int_{E\cap B_2} g)$$

The set C may be treated in a similar manner.

Finally, by Theorem 9.1, we have

$$\int_E f+g = \int_{E\cap A}(f+g) + \int_{E\cap B_1}(f+g) + \int_{E\cap B_2}(f+g)$$

$$= \int_{E\cap A} f + \int_{E\cap A} g + \int_{E\cap B_1} f + \int_{E\cap B_1} g + \int_{E\cap B_2} f + \int_{E\cap B_2} g$$

$$= \int_E f + \int_E g$$

9.3. THEOREM. If f is any integrable function and if $m(E) = 0$, then $\int_E f = 0$.

PROOF. If f is a simple function, then by definition

$$\int_E f = \sum_{i=1}^{n} a_i\, m(E\cap E_i) = 0 \;, \quad \text{because}$$

$m(E\cap E_i) \leq m(E) = 0$ for all i.

If f is any nonnegative integrable function corresponding to Definition 8.12 we obtain $\int_E f = \lim \int_E f_n = 0$ because of the above result.

If f is any integrable in general case, then by Definition 8.13

$$\int_E f = \int_E f^+ - \int_E f^- = 0$$

We have shown that the integral of any integrable function is insignificant over a set of measure zero. From that concept, we next discuss more on what we have considered in Theorem 8.14.

9.4. THEOREM. <u>If</u> f <u>is integrable on</u> E <u>and if</u> $f \geq 0$ a.e. <u>on</u> E, <u>then if for all</u> $A \subset B \subset E$, <u>we have</u>

$$\int_A f \geq 0 \quad \text{and} \quad \int_A f \leq \int_B f .$$

PROOF.

$$F = \{x : f(x) < 0\} \ , \ \text{then} \ m(F) = 0,$$

and $\int_F f = 0$ by 9.3.

It follows Theorem 8.14, the results are true on the measurable set $E - F$.

Now, for every $A \subset E$, we show that $\int_A f \geq 0$.

From the fact that $A = (A \cap (E - F)) \cup (A \cap F)$, then

$$\int_A f = \int_{A \cap (E-F)} f + \int_{A \cap F} f \geq 0 .$$

Now, for all $A \subset B \subset E$, we have

$$B = A \cup (B - A) \cap E ,$$

thus,

$$\int_B f = \int_A f + \int_{(B-A) \cap E} f$$

Since $\int_{(B-A) \cap E} f \geq 0$ by the first result, then

$$\int_B f \geq \int_A f \quad \text{for every } A \subset B \subset E$$

completing the proof.

9.5. COROLLARY. <u>If</u> f <u>and</u> g <u>are integrable on</u> E <u>and if</u> $f \leq g$ <u>a.e. on</u> E, <u>then</u> $\int_E f \leq \int_E g$.

PROOF. By the theorem, since $g - f \geq 0$ a.e. then $\int_E (g - f) \geq 0$; the result now follows from Theorem 9.2.

Next we extend the validity of Theorem 8.8 to the general case of the class of integrable functions.

9.6. THEOREM. <u>If</u> $0 \leq f \leq g$ <u>a.e. on</u> E, <u>if</u> f <u>is measurable, and if</u> g <u>is integrable on</u> E, <u>then</u> f <u>is integrable on</u> E.

PROOF. Let $A = \{x:f(x) > 0\}$ and $B = \{x:g(x) > 0\}$. Since $f \leq g$, if $x \varepsilon A$, then $g(x) \geq f(x) \geq 0$, so $x \varepsilon B$, or $A \subset B$. Since g is integrable, then there exists a nondecreasing sequence $\{g_n\}$ of nonnegative simple functions, each of which is integrable on E (i.e. for each n, $\{x:g_n(x) > 0\}$ has finite measure). Then by Theorem 9.3, B is a union of a countable number of measurable sets of finite measure, and so is A. Since f is a measurable function, by Theorem 9.2, there exists a nondecreasing sequence $\{f_n\}$ of nonnegative, simple functions converging pointwise to f. Since A is the union of a countable class of measurable sets of finite measure, then by 9.3, for each n, $\{x:f_n(x) > 0\}$ has finite measure, in other words, f_n is an integrable simple function. Thus by corollary 9.5, we have

$$\int_E f_n \leq \int_E g < \infty \quad \text{for all n.}$$

Hence,

$$\lim \int_E f_n < \infty. \quad \text{So f is integrable on E.}$$

▮

9.7. COROLLARY. <u>A measurable function</u> f <u>is integrable on</u> E <u>if and only if</u> $|f|$ <u>is integrable on</u> E.

PROOF. Now,

$$|f| = f^+ + f^- .$$

By Definition 8.13 if f is integrable on E, then so are f^+ and f^-. Hence, by Theorem 9.2, $|f|$ is integrable on E.

Conversely, suppose $|f|$ is integrable on E. Now by Theorem 6.1, f^+ is a measurable function if f is. Moreover, since $0 \leq f^+ = |f| - f^- \leq |f|$, it follows from Theorem 9.6 that f^+ is integrable on E. A similar argument applies to f^- . Therefore f is integrable on E , concluding the proof.

∎

The same argument as Theorem 5.3 in Chapter II, the class of integrable functions on E is a vector lattice.

9.8. COROLLARY. <u>If</u> f <u>is a measurable function</u>, <u>if</u> g <u>is integrable on</u> E, <u>and if</u> $|f| \leq g$ a.e. <u>on</u> E, <u>then</u> f <u>is integrable on</u> E.

PROOF. By Theorem 9.6, $|f|$ is integrable on E, hence by Corollary 9.7, f is also integrable there.

∎

9.9. COROLLARY. <u>If</u> $f = g$ a.e. <u>on</u> E <u>and if</u> g <u>is integrable on</u> E, <u>then</u> f <u>is integrable on</u> E <u>and</u>

$$\int_E f = \int_E g .$$

PROOF. Since $0 \leq f^+ \leq g^+$ and $0 \leq f^- \leq g^-$ a.e., by Theorem 9.6, f^+ and f^- are integrable on E. Therefore f is integrable on E by definition. The equality of the integrals now follows from 9.5.

∎

9.10. THEOREM. <u>If</u> f <u>is integrable on</u> E, <u>then</u>

$$\left| \int_E f \right| \leq \int_E |f| \ .$$

PROOF. Recall that $|f| = f^+ + f^-$. Since $\int_E f^+ + \int_E f^-$ are both nonnegative and $f = f^+ - f^-$, from 9.5, we have

$$\left| \int_E f \right| = \left| \int_E f^+ - \int_E f^- \right|$$

$$\leq \max\{ \int_E f^+ , \int_E f^- \} \leq \int_E f^+ + \int_E f^-$$

$$= \int_E |f| \quad , \text{ completing the proof.}$$

∎

9.11. THEOREM. <u>If</u> $f \geq 0$ a.e. <u>on</u> E <u>and if</u> $\int_E f = 0$, <u>then</u> $f = 0$ a.e. <u>on</u> E.

PROOF. Let $A = \{x \in E : f(x) > 0\}$. For each integer n, let $A_n = \{x \in E : f(x) > \frac{1}{n}\}$; then $\{A_n\}$ is a nondecreasing sequence of measurable sets, thus $\lim A_n = A$ because of

$$\{x \in E : f > \lim_n (\tfrac{1}{n}) = 0\} = A \ .$$

Thus $\lim_n m(A_n) = m(A)$. If $m(A) > 0$, there is an n such that $m(A_n) > 0$. By 9.4 and 9.5, since $A_n \subset E$ and $f \geq 0$ a.e. then

$$\int_E f \geq \int_{A_n} f \geq \int_{A_n} \frac{1}{n} = \frac{1}{n} m(A_n) > 0 \ .$$

This is a contradiction. So $m(A) = 0$ or $f = 0$ a.e. on E.

9.12. COROLLARY. *If* f *and* g *are both integrable on* E *and if* $\int_A f = \int_A g$ *for every* $A \subset E$, *then* $f = g$ a.e. *on* E.

PROOF. Let $E_1 = \{x \varepsilon E; f(x) \geq g(x)\}$.

Then by hypothesis

$$\int_{E_1} (f - g) = 0 .$$

Hence, by the theorem, $g = g$ a.e. in E_1 . Similarly, $f = g$ a.e. in $E_2 = \{x \varepsilon E : f(x) \leq g(x)\}$, and

$\int_{E_2} (f - g) = 0$. Now, $\int_E (f - g) = \int_{E_1} (f - g) + \int_{E_2} (f - g) = 0$

Hence, by the theorem $f - g = 0$ a.e. on E.

∎

10. SOME CONVERGENCE THEOREMS

In this section we present some of the most important convergence theorems belonging to the general integral. The first of these theorems is often referred to as the Lebesque Monotone Convergence theorem for integrals.

10.1. THE LEBESQUE MONOTONE CONVERGENCE THEOREM. *Let* $\{f_n\}$ *be a nondecreasing sequence of nonnegative functions, that are integrable on* E *and let* f *be a function such that* $\lim f_n = f$ a.e. *on* E. *Then* f *is integrable on* E *if and only if* $\lim \int_E f_n < \infty$, *and if this is the case, then*

$$\lim \int_E f_n = \int_E f .$$

PROOF. For each n, f_n is a nonnegative integrable function on E, then by definition, there exists a nondecreasing sequence of nonnegative integrable simple functions $\{f_{nk}\}_{k=1}^{\infty}$ converging pointwise to f_n:

$$\lim_k f_{nk}(x) = f_n(x) \quad , \text{ gives the array}$$

$$\begin{array}{cccccccc}
f_{11} & f_{12} & \cdots & f_{1k} & \cdots & \to & f_1 \\
f_{21} & f_{22} & \cdots & f_{2k} & \cdots & \to & f_2 \\
\vdots & \vdots & & \vdots & & & \vdots \\
f_{n1} & f_{n2} & \cdots & f_{nk} & \cdots & \to & f_n \\
\vdots & \vdots & & \vdots & & & \vdots
\end{array}$$

We define $h_i = \max_{n \leq i} f_{ni}$

It is clear that $\{h_i\}$ is a nondecreasing sequence of nonnegative, integrable, simple functions, and since $\{f_n\}$ is a sequence of nondecreasing functions, for each n and $n \leq i$

$$f_{nk} \leq h_i \leq \max_{n \leq i} f_n = f_i \tag{1}$$

So when $i \to \infty$, for all n, we have

$$f_n \leq \lim h_i \leq f \quad \text{a.e.}$$

By hypothesis, then we have $\lim f_n = \lim h_i = f$ a.e.

Corollary 9.9 takes care of the exceptional point-set of measure zero. By Definition 8.12, if $\lim \int_E h_i < \infty$ then f is integrable on E and

$$\lim \int_E h_i = \int_E f \quad .$$

However, from (1) above, and 9.5, we have

$$\int_E h_i \leq \int_E f_i \qquad \text{for all } i.$$

Thus

$$\int_E f = \lim \int_E h_i \leq \lim \int_E f_n \tag{2}$$

Now, the converse of the theorem is that if $\lim \int_E f_n < \infty$, then we have f is integrable on E by (2).

Moreover, since $\{f_n\}$ is a nondecreasing sequence, then

$$f_n \leq f \quad \text{a.e.} \quad \text{for each } n.$$

Thus $$\int_E f_n \leq \int_E f \quad \text{for all } n.$$

Then

$$\lim \int_E f_n \leq \int_E f \tag{3}$$

Thus, if f is integrable on E, then $\lim \int_E f_n < \infty$.

Together with (2) and (3), this completes the proof.

∎

The following result is an immediate consequence of Theorem 10.1 and is sometimes more convenient to apply.

10.2. COROLLARY. *Let $\{f_n\}$ be an a.e. nonnegative sequence of functions, each of which is integrable on E. If $\sum_{n=1}^{\infty} f_n(x) = f(x)$ a.e. on E, then f is integrable on E if and only if $\sum_{n=1}^{\infty} \int_E f_n$ is convergent, and if this is the case,*

$$\int_E f = \sum_{n=1}^{\infty} \int_E f_n \quad .$$

PROOF. Let

$$h_i = \sum_{n=1}^{i} f_n \quad .$$ Then $\{h_i\}$ is a nondecreasing sequence of nonnegative functions, each of which is integrable on E. Also,

$$\lim h_i = \sum_{n=1}^{\infty} f_n = f \; .$$

By the theorem, we have

$$\lim \int_E h_i = \int_E \lim h_i = \int_E \Big(\sum_{n=1}^{\infty} f_n \Big) = \int_E f \quad .$$

That is,

$$\sum_{n=1}^{\infty} \int_E f_n = \int_E f \quad .$$

∎

10.3. COROLLARY. Let $\{f_n\}$ be an a.e. nonincreasing sequence of functions, each of which is integrable on E, and assume that $\int_E f_1 < \infty$. If

$$\lim f_n = f \text{ a.e., then } \lim \int_E f_n = \int_E f .$$

PROOF. Now, $\{f_1 - f_n\}$ is a nondecreasing sequence of nonnegative functions, each of which is integrable on E, and

$$\lim_n (f_1 - f_n)(x) = (f_1 - f)(x) \text{ a.e. on E.}$$

Thus by the theorem

$$\lim_n \int_E (f_1 - f_n) = \int_E (f_1 - f)$$

or

$$\lim \int_E f_n = \int_E f .$$

In Chapter II the approach of the Monotone Convergence theorem is a consequence of Levi's Theorem 6.2. In this chapter, we approach directly from the definition of nonnegative integrable functions. But, the important purpose of this theorem is the same thing, that is to find out the conditions for which a limit function of a sequence of integrable functions is integrable.

Next, we turn to another important theorem. If f is integrable, then f is integrable over any measurable set

belonging to σ-ring of measurable sets $\mathcal{M}$. We will extend the Theorem 9.1 which is still valid in case of a sequence of disjoint measurable sets.

10.4. THEOREM. <u>Let $\{E_n\}$ be a sequence of disjoint measurable sets, and let</u> $E = \bigcup_{n=1}^{\infty} E_n$. <u>Let f be a nonnegative measurable function that is integrable on each set</u> E_n . <u>Then, f is integrable on E if and only if</u>

$$\sum_{n=1}^{\infty} \int_{E_n} f < \infty ,$$

<u>and if this is the case,</u>

$$\int_E f = \sum_{n=1}^{\infty} \int_{E_n} f .$$

PROOF. For each i, let $A_i = \bigcup_{n=1}^{i} E_n$ then $\lim A_i = E$.

Next, let the sequence $\{f_i\}$ be defined by

$$f_i(x) = \begin{cases} f(x) & \text{if } x \text{ in } A_i \\ 0 & \text{otherwise} \end{cases}$$

Then $\{f_i\}$ is a nondecreasing sequence of nonnegative functions, each of which is integrable on E because $A_i \subset A_{i+1}$ for each i and if f is integrable on E, so is f_i. From 9.1, we have

$$\int_E f_i = \int_{A_i} f = \sum_{n=1}^{i} \int_{E_n} f \quad .$$

By monotone convergence theorem, we have

$$\int_E f = \lim \int_E f_i = \sum_{n=1}^{\infty} \int_{E_n} f \quad .$$

Later on, we will find other conditions which guarantee that

$$\int_E \lim f_n = \lim \int_E f_n \quad .$$

However, this will depend on the manner in which $\{f_n\}$ converges to f. In the case of a.e. convergence, the strongest statement that holds in general is the following one.

10.5. LEMMA (FATOU). <u>Let</u> $\{f_n\}$ <u>be a sequence of nonnegative integrable functions on</u> E <u>satisfying</u> $\underline{\lim}_n f_n(x) = f(x)$ a.e.

<u>If</u> $\underline{\lim}_n \int_E f_n < \infty$, <u>then</u> f <u>is integrable and moreover</u>

$$\int_E f \leq \underline{\lim}_n \int_E f_n \quad .$$

PROOF. Recall that $\underline{\lim}\, f_k (\lim_j (f_k \wedge f_{k+1} \wedge \ldots \wedge f_j))$

Let $g_n(x) = \inf\{f_i(x) : n \leq i\}$ then $g_n \leq f_n$ and the sequence $\{g_n\}$ is nondecreasing. From Corollary 9.5

$$\int_E g_n \leq \int_E f_n \qquad \text{for all } n \quad ,$$

it follows that

$$\int_E g_n \leq \underline{\lim}_{n} \int_E f_n < \infty \quad ,$$

since $\lim g_n(x) = \underline{\lim}_{n} f_n(x) = f(x)$ a.e. on E.

It follows from Lebesque monotone convergence theorem that f is integrable on E and

$$\int_E f = \lim \int_E g_n \leq \underline{\lim}_{n} \int_E f_n \quad .$$

∎

Using the fact that $\overline{\lim}_{n} f_k = -\underline{\lim}(-f_k)$ we obtain a result similar to Fatou's lemma for the limit superior.

10.6 LEMMA. *Let* $\{f_n\}$ *be a sequence of nonnegative, integrable functions on* E *with* $\overline{\lim}_{n} f_n(x) = f(x)$ a.e. *and let* $\overline{\lim}_{n} \int_E f_k > -\infty$, *then* f *is integrable on* E *and*

$$\int_E f \geq \overline{\lim}_{n} \int_E f_n \quad .$$

From the two previous lemmas we obtain the following important theorem. In order to have

$$\int_E \lim f_n = \lim \int_E f_n \quad ,$$

the conditions of the Lebesque Monotone Convergence Theorem are difficult. In the following theorem, we have better conditions to get our purpose.

10.7. LEBESQUE DOMINATED CONVERGENCE THEOREM. Let $\{f_n\}$ be a sequence of integrable functions on E such that $\{f_k(x)\}$ converges a.e. on E and for all k, $|f_k| \leq g$ a.e. on E for some integrable function g on E. If $\lim f_k(x) = f(x)$ a.e. on E, then f is integrable and

$$\lim \int_E f_k = \int_E f \quad .$$

PROOF. To prove this theorem, we use a similar argument from Lebesque Dominated Theorem 6.8 of Chapter II.

CONCLUSIONS: CONNECTIONS OF THE TWO APPROACHES

The purpose of this thesis has been to help the reader round out his knowledge of integration and measure theory; to understand the approach via linear functionals as well as by the more standard approach via measure theory. Each subject is developed fairly completely from these two points of view although the procedures are quite different in each case.

In Chapter II, we developed a theory of measurable functions and their integrals without any use of a theory of measure. We then used this theory of integrable functions in order to develop a theory of measurable functions and measurable sets. On the other hand, in the last chapter, we began with an arbitrary measure space from which we constructed, first a theory of measure in terms of set-functions, and then a theory of measurable functions. Finally we used this theory to develop a theory of integrable functions and their integrals.

The questions that we must now ask ourselves are the following: Suppose we take the measure space $(X,\mathcal{M},m)$ determined by the integral as explained in the first approach, and use the methods in the third chapter. Will the resulting classes of measurable functions and integrable functions then be the same as those previously obtained in the second chapter? And will the values of the integrals be the same? We will now demonstrate that both of these questions are answered affirmatively.

In order to keep our ideas clear, we must distinguish between the notions of integrability and measurability for functions in the sense of the two chapters. A function that is measurable and integrable in the sense of Chapter II will be called D-<u>measurable</u> and D-<u>integrable</u>, respectively. Likewise, a function measurable and integrable in the sense of Chapter III will be called m-measurable and m-integrable, respectively. The class $\mathcal{M}$ of measurable subsets of X in the first approach will be taken as the class of measurable sets in the second approach.

According to Proposition 7.9 in Chapter II, if satisfies Stone's Axiom ($f \in \mathcal{S}$ implies $1 \wedge f \in \mathcal{S}$), then the constant function is a D-measurable function and we can say that X is a measurable set. In Chapter III, the theory says that X is in (Theorem 4.5; Chapter III). Thus, in order to prove that the theory of integration developed in Chapter III is equivalent to the theory developed in Chapter III, we need to remember that X is a measurable set in Chapter III (since in Chapter II we accepted Stone's axiom, Proposition 8.3, Chapter II).

In the following theorem, we will show that the classes of D-measurable functions and m-measurable functions are equivalent.

1. THEOREM. <u>Let</u> $f:X \to R$. <u>Then</u> f <u>is</u> <u>a</u> D-<u>measurable</u> <u>function</u> <u>if</u> <u>and</u> <u>only</u> <u>if</u> f <u>is</u> m-<u>measurable</u>.

PROOF. We use the criteria for m-measurability expressed in Theorems 5.6 and 5.7 of Chapter III. First, suppose f is a D-measurable function. For any given c in R let $E = \{x : f(x) > c\}$. Define the sequence of D-measurable functions

$$f_n = n(f - (f \wedge \underline{c}))$$

where $\underline{c}$ is the constant function with value c for all $x \in X$. Since $X \in \mathcal{M}$, then $\underline{1}$ is a D-measurable function and so is $\underline{c}$. Thus $f \wedge \underline{c}$, and hence f_n, are D-measurable functions. Let $g_n = \underline{1} \wedge f_n$. We will show that $\{g_n\}$ is a nondecreasing sequence of functions which converges to χ_E (the characteristic function of the set E).

If $x \in E$, we have $f_n(x) = n(f(x) - c)$ tends to $+\infty$ as n tends to ∞. Thus $g_n(x) = 1 \wedge f_n(x) = 1$ for n large enough; thus, $g_n(x)$ tends to 1. If x is in $X - E$, then $f(x) \wedge c = f(x)$, and $g_n(x) = f_n(x) = 0$ for all n. Now, since g_n is D-measurable and $\lim g_n = \chi_E$ is also D-measurable by Proposition 7.6 (Chapter II), then E is a measurable set by definition. To complete the proof that f is m-measurable. By Theorem 5.6 (Chapter III), E is a measurable set, and by 5.6(1), 5.6(4) and 5.7 (Chapter III), f is a m-measurable function.

Conversely, suppose that f is m-measurable. It follows from Theorem 6.1 in Chapter III that f^+ and f^- are m-measurable functions. If we show that f^+ and f^- are D-measurable, it

follows f is D-measurable (Proposition 7.5 in Chapter II). Hence it suffices to consider the case in which $f \geq 0$. Then we complete the proof by constructing a nondecreasing sequence $\{g_n\}$ of D-measurable functions which converges to f; by Proposition 7.6 (Chapter II), f is then D-measurable.

Let $E = \{x:f(x) = 0\}$ and $F = \{x:f(x) = \infty\}$. Since f is m-measurable and X is a measurable set, by Theorems 5.6 and 5.7 (Chapter III), $f^{-1}([0,\infty])$ is a measurable set. Since $E = X - \{x:f(x) > 0\}$ is a measurable set as in the proof of Theorem 5.7 (Chapter III) we have that

$$f^{-1}(+\infty) = X - \bigcup_{n=1}^{\infty} f^{-1}\{[0,n]\}$$

is also a measurable set. The same holds true of F. Now, choose $a > 1$ and consider the sets

$$E_i(a) = \{x:a^i < f(x) \leq a^{i+1}\}, \quad i=0, 1, 2,...$$

By Theorem 5.6 (Chapter III), these sets are measurable. Moreover every point of X belongs to exactly one of the sets

$$E, F, E_o(a), E_1(a), E_{-1}(a), E_2(a), E_{-2}(a), \ldots .$$

Define the function h as follows:

$$h(x) = \begin{cases} \infty, & \text{if } x \in F \\ 0, & \text{otherwise.} \end{cases}$$

Then h is a D-measurable function because for every function g which is D-integrable, $(-g) \vee (h \wedge g) = -g$ is D-integrable by definition. Let h_i be the characteristic function of the measurable set $E_i(a)$ so that h_i is D-measurable for all i by definition. Then the function

$$h + \sum_{i=-n}^{n} a^i h_i$$

is a D-measurable function (Proposition 7.5, Chapter II) for each n. This is a nondecreasing sequence of functions which converges to a limit function f_a as n tends to ∞, and this limit function f_a is D-measurable. Moreover f_a has the values 0, $+\infty$, and a^i on E, F, and $E_i(a)$, respectively.

Next, we choose a sequence $\{a_n\}$ of values for a converging decreasing to 1. We also choose some fixed $\delta > 1$ and let $a_n = \delta^{\alpha_n}$, where $\alpha_n = 2^{1-n}$. We see that

$$\alpha_{n+1} = 2^{1-(n+1)} = 2^{1-n-1} = \alpha_n \, 2^{-1} .$$

Thus $a_{n+1} = a_n^{1/2}$, or $a_{n+1}^{2i} = a_n^i$. Since $a_n > 1$ for all n, then a_{n+1}^{2i+1} is between a_n^i and a_n^{i+1} ($= a_{n+1}^{2i+2}$). This means that

$$E_i(a_n) = E_{2i}(a_{n+1}) \cup E_{2i+1}(a_{n+1}) .$$

Now, let $g_n = f_{a_n}$. Then $\{g_n\}$ is a nondecreasing sequence of functions since $\{f_{a_n}\}$ is nondecreasing by definition. Also, for all $x \varepsilon E \cup F$ and for all n, $g_n(x) = f(x)$. Finally we have that for any $x \varepsilon E_i(a_n)$

$$0 < f(x) - g_n(x) < a_n^{i+1} - a_n^i = a_n^i(a_n - 1) < f(x)(a_n - 1) .$$

Therefore, when $\{a_n\}$ converges to 1, we see that $\{g_n\}$ converges pointwise to f. Thus, f is a D-measurable function as claimed, completing the proof.

∎

Because of the previous theorem, from now on we will simply refer to the class of D-measurable fucntions or the class of m-measurable functions as <u>the class of measurable functions</u>.

Next, we will consider the question of integrability.

2. THEOREM. *Let* $f:X \to R$ *be a measurable function. Then* f *is* D-*integrable if and only if* f *is* m-*integrable; and in this case both integrals have the same value*.

PROOF. We only prove the theorem for the case $f \geq 0$. The general case then follows by using the decomposition $f = f^+ - f^-$.

For convenience, we denote by $\mathscr{L}(D)$ and $\mathscr{L}(m)$ the classes of D-integrable and m-integrable functions, respectively. And we denote by $I(f;D)$ and $I(f;m)$ the integrals of f when $f \varepsilon \mathscr{L}(D)$ or $f \varepsilon \mathscr{L}(m)$, respectively.

Suppose that $f \geq 0$ and that either $f \in \mathcal{L}(D)$ or $f \in \mathcal{L}(D)$. Then the function f is measurable and we can construct the functions h, h_i, f_a, and g_n just as in the proof of Theorem 1 give above. Either $f \in \mathcal{L}(D)$ or $f \in \mathcal{L}(m)$, in order the integral of f exists, it implies the $m(F) = 0$, where $F = \{x:f(x) = \infty\}$; thus by Corollary 6.5 (Chapter II) and Theorem 9.3 (Chapter III), f is both D-integrable and m-integrable and $I(h;D) = I(h;m) = 0$. We also have that $0 \leq h_i \leq a^{-i}f$. Since h_i is measurable, f is D-integrable implies that h_i is D-integrable. Likewise, f is m-integrable implies that h_i is m-integrable. In either case, since h_i is a characteristic function and $m(E_i(a)) < +\infty$, then according to definitions 9.1 (Chapter II) and 8.5 (Chapter III), h_i belongs to both $\mathcal{L}(D)$ and $\mathcal{L}(m)$. Moreover in either case of the hypothesis on f,

$$I(h_i;D) = I(h_i;m) = m(E_i(a)) .$$

Next, let $H_n = h + \sum_{i=-n}^{n} h_i$. Then for each n, H_n belongs to both $\mathcal{L}(D)$ and $\mathcal{L}(m)$. Moreover, $\{H_n\}$ is a nondecreasing sequence of functions that converges to f_a, and $f_a \leq f$. For every n, $I(H_n;D) = I(H_n;m)$, and the sequence $\{I(H_n;D)\}$ is bounded above:

by $I(f;D)$ if $f \in \mathcal{L}(D)$; and by $I(f;m)$, if $f \in \mathcal{L}(m)$.

By the Monotone Convergence Theorem (of either case) we can conclude that f_a belong to both $\mathcal{L}(D)$ and $\mathcal{L}(m)$, and that

$$I(f_a;D) = I(f_a;m) \quad .$$

Finally, replace a by a_n so that f_a becomes g_n, and we have that the nondecreasing sequence $\{g_n\}$ of nonnegative integrable functions converges to f. By the exact same argument we then have that f is integrable in both senses, and that

$$I(f;D) = I(f;m) \quad .$$

This concludes the proof.

We have come full circle as was claimed in the introduction to this section.

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